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INNOVATION DYNAMICS, PATENTS, AND DYNAMIC-ELASTICITY TESTS FOR THE PROMOTION OF PROGRESS

*John M. Golden**

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I. INTRODUCTION

Much recent debate over patents has focused on their ability to act as brakes as well as stimulants to innovation. Theoretical studies have described how patents can impede innovation, whether by generating an "anticommons"[1] or by otherwise taxing, complicating, or outright blocking the efforts of later developers.[2] In books with such foreboding titles as *Patent Failure*[3] and *Innovation and Its Discontents*,[4] commentators have argued that these undesirable theoretical capacities are being realized and that patents, government grants meant to stimulate technological progress, might be instead doing much to hold it back.[5]

The recognized tension between patents' capacities to stimulate and to slow technological progress highlights an important point. Whether patents serve their constitutionally specified role of "promot[ing] . . . Progress"[6] depends significantly on the dynamics through which that progress occurs. For any given field of technology, these dynamics determine whether patents' innovation-stimulating effects outweigh their innovation-impeding effects. Moreover, the trajectory of technological progress that these dynamics generate can have significant implications for patent policy.[7] Papers by Polk Wag-

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1. Michael A. Heller & Rebecca S. Eisenberg, *Can Patents Deter Innovation? The Anticommons in Biomedical Research*, 280 SCIENCE 698, 698–99 (1998) (describing "two mechanisms" that can generate an "anticommons" in which "multiple owners each have a right to exclude others from a scarce resource and no one has an effective privilege of use").

2. *See, e.g.*, STEVEN SHAVELL, FOUNDATIONS OF ECONOMIC ANALYSIS OF LAW 148 (2004) ("[T]he broader the protection granted to first innovations, the lower the incentives of others to generate second innovations."); Robert P. Merges & Richard R. Nelson, *On the Complex Economics of Patent Scope*, 90 COLUM. L. REV. 839, 843 (1990) ("[T]he notion of a patent's social costs should include its potential to reduce competition in the market for improvements to the patented technology."). *See generally* SUZANNE SCOTCHMER, INNOVATION AND INCENTIVES 127 (2004) ("The problem introduced for incentive mechanisms is how to make sure that earlier innovators are compensated for their contributions, while ensuring that later innovators also have an incentive to invest.").

3. JAMES BESSEN & MICHAEL J. MEURER, PATENT FAILURE: HOW JUDGES, BUREAUCRATS, AND LAWYERS PUT INNOVATORS AT RISK 141 (2008) ("[W]e can safely conclude that during the late 1990s, the aggregate costs of patents exceeded the aggregate private benefits of patents for United States public firms outside the chemical and pharmaceutical industries.").

4. ADAM B. JAFFE & JOSH LERNER, INNOVATION AND ITS DISCONTENTS: HOW OUR BROKEN PATENT SYSTEM IS ENDANGERING INNOVATION AND PROGRESS, AND WHAT TO DO ABOUT IT 2 (2004) ("[T]he patent system . . . is generating waste and uncertainty that hinders and threatens the innovative process.").

5. DAN L. BURK & MARK A. LEMLEY, THE PATENT CRISIS AND HOW THE COURTS CAN SOLVE IT 30 (2009) ("At least three major economic studies in the last five years have suggested that the patent system may actually do more harm than good to innovation . . . .").

6. U.S. CONST. art. I, § 8, cl. 8.

7. Policymakers and their advisors have long recognized the importance of having some sense of the likely trajectory of technological progress. *Cf.* SUBCOMM. ON TECH., NAT'L

ner and Dennis Karjala suggest that, if the growth of technological information is fundamentally exponential in time, the extent to which patents impede patentable technologies' use and availability might be less of a concern than if the growth of technological information is essentially linear.[8] More generally, commentators commonly assume one or another form for "natural" patent dynamics when they suggest that the significant growth in the number of patent applications and issued patents over the last few decades indicates a fundamental problem, rather than plausibly reflecting such presumed social goods as increased technological progress or increased success in the patent system's promotion of technological disclosure as an alternative to secrecy.[9]

But any assumptions that we know the "natural" form of technological progress in one or another field, or as a whole, seem prema-

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RES. COMM., TECHNOLOGICAL TRENDS AND NATIONAL POLICY vii (1937) ("The large number of inventions made every year shows no tendency to diminish . . . . No cessation of social changes due to invention is to be expected.").

8. *Compare* R. Polk Wagner, *Information Wants To Be Free: Intellectual Property and the Mythologies of Control*, 103 COLUM. L. REV. 995, 1018 (2003) (postulating that information grows exponentially with time), *with* Dennis S. Karjala, *Does Information Beget Information?*, 2007 DUKE L. & TECH. REV. 1, 8–10 (arguing that Wagner's use of an exponential-growth model is substantially responsible for Wagner's conclusion that more control over information is likely to lead to the generation of more freely available information). *But cf.* Jonathan Huebner, *A Possible Declining Trend for Worldwide Innovation*, 72 TECHNOLOGICAL FORECASTING & SOC. CHANGE 980, 980 (2005) (contesting an alleged "general consensus that technology is advancing exponentially").

9. *See, e.g.*, BURK & LEMLEY, *supra* note 5 at 22 ("Even if the world is more innovative than it used to be, we doubt it is four times more innovative than . . . in the 1980s, or . . . nearly twelve times as innovative as the 1870s . . . . The more logical explanation is that it is simply easier to get a patent today than it used to be . . . ."). More sophisticated analyses have not cited raw growth rates as evidence of patent numbers' disconnect from actual rates of technological progress, but have instead focused on an apparent discontinuity in the growth rate of U.S. patent applications in the early 1980s, which approximately coincided with the creation of the U.S. Court of Appeals for the Federal Circuit. *See* Bronwyn H. Hall, *Exploring the Patent Explosion*, 30 J. TECH. TRANSFER 35, 46 (2005) [hereinafter Hall, *Exploring*] (reporting "clear evidence of a structural shift to a higher growth rate in overall patenting in the United States between 1983 and 1984"); Bronwyn H. Hall, *Patents and Patent Policy*, 23 OXFORD REV. ECON. POL'Y 568, 578 (2007) (associating "a structural break in 1984" in U.S. patent numbers with "changes in the US patent system"); F.M. Scherer, *The Political Economy of Patent Policy Reform in the United States* 28 (John F. Kennedy Sch. Of Gov't Faculty Research Working Papers Series, Paper No. RWP07-042, 2007), *available at* http://ssrn.com/abstract=963136 (associating "a distinct and statistically significant break in [patent numbers] at the year 1983" with "the creation of the [Federal Circuit]"). At least two commentators have argued, however, that the acceleration of patenting during this period and its aftermath largely reflected "increases in innovation and[] improvements in the management of R&D." Hall, *Exploring*, *supra*, at 35; *see* Samuel Kortum & Josh Lerner, *Stronger Protection or Technological Revolution: What Is Behind the Recent Surge in Patenting?*, 48 CARNEGIE-ROCHESTER CONF. SERIES ON PUB. POL'Y 247, 251–52 (1998) ("tentatively conclud[ing]" that "the increase in patent activity [in the United States] seems to be a consequence of a world-wide increase, along with a recent improvement in the relative performance of U.S. inventors," and positing "that much of the increase may be due to improvements in the management or automation of the innovation process").

ture. Like the understanding of entrepreneurial dynamics,[10] the understanding of innovation dynamics appears to be in its infancy. Indeed, "infancy" is perhaps too generous a term. Understanding of innovation dynamics might be better characterized as embryonic.

Prior theoretical work has often focused on how patents or other incentives or impediments to innovation might affect discrete decisions about whether, and with what level of resource commitment, to pursue a particular line of research.[11] Although such decision models can be instructive, their isolated-decision-point structure tends to result in their neither suggesting a particular trajectory for progress nor directing empirical work in a way likely to lead to deeper understanding of innovation dynamics.

This Article steps into the breach by presenting a fluid-mechanics-inspired model for innovation dynamics. The model provides a relatively straightforward framework for considering the interaction between potential "pushes" that tend to accelerate innovative progress and potential "drags" that tend to decelerate it. Within the model, a particular policy could contribute to both progress-promoting pushes and progress-impeding drags. Thus, for example, a move toward construing patents' scope more broadly can act as a push by increasing patent-associated rewards for innovators.[12] On the other hand, that same move can also act as a drag by increasing follow-on innovators' costs for reviewing and clearing patent rights owned by others.[13] The model's combination of push and drag terms in one dynamic equation makes the model a good vehicle for considering this double-headed character of patents as both potential stimulants and potential brakes to innovation.

The model also provides an opportunity to study positive and negative feedbacks in technological development. Through incorpora-

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10. *See* JOSH LERNER, BOULEVARD OF BROKEN DREAMS: WHY PUBLIC EFFORTS TO BOOST ENTREPRENEURSHIP AND VENTURE CAPITAL HAVE FAILED — AND WHAT TO DO ABOUT IT vii (2009) (observing that "economists have turned only recently to the question of how to boost entrepreneurship").

11. *See, e.g.*, ROBERT PATRICK MERGES & JOHN FITZGERALD DUFFY, PATENT LAW AND POLICY: CASES AND MATERIALS 696–99 (4th ed. 2007) (presenting a "two-step decision model" "[t]o explore the effects of the nonobviousness standard on the incentives facing a prospective researcher/inventor" (emphasis omitted)); SCOTCHMER, *supra* note 2, at 136–41 (presenting a model for "[t]he strategic environment of licensing" in which actors make discrete decisions in accordance with expected payoffs).

12. *See* SCOTCHMER, *supra* note 2, at 97 ("[I]ntellectual property is an important incentive mechanism."); Merges & Nelson, *supra* note 2, at 871 (discussing how, according to Edmund Kitch's "'prospect theory' of patent rights," a broad, early patent "allows 'breathing room' for the inventor to invest in development").

13. *See* Merges & Nelson, *supra* note 2, at 870 (contending that "broad patents could discourage much useful research" in follow-on innovation); *cf.* Ariad Pharms., Inc. v. Eli Lilly & Co., 598 F.3d 1336, 1353 (Fed. Cir. 2010) (en banc) (acknowledging that denial of patent protection for "research hypotheses . . . possibly results in some loss of incentive," but countering that "claims to research plans also impose costs on downstream research, discouraging later invention").

tion of such feedback effects, the model generates a rich array of functional behaviors from a relatively small number of terms. Distinct circumstances and time scales correspond to different forms of functional behavior, such as exponential, linear, or higher-power-law growth. A given form of functional behavior is commonly transient, ultimately undercutting its own existence by causing a previously negligible form of feedback to become significant.

The model thus confirms common intuitions that processes of innovation are highly context-dependent. More specifically, the model suggests that there is no single "natural" form for the time-dependent behavior of technological progress. Indeed, within this model, the exponential growth posited by Wagner[14] prevails under only a relatively exceptional and unstable set of circumstances. In contrast, for a wide variety of circumstances, some form of power-law growth seems likely to prevail. But the particular exponent characteristic of such power-law growth is not a universal constant. Instead, it depends strongly on the nature of the dominant pushes and drags.

Moreover, elaborations of the model's basic form show how a period of rapid technological progress can end when progress encounters a technological "wall" or "bottleneck." Such an impediment might eventually be overcome or circumvented, leading to a resumption of rapid growth that continues until a new wall or bottleneck is encountered. Over very long time scales, the resulting trajectory of technological progress might feature a series of *S*-like shapes marking successive cycles of acceleration and deceleration.

Such complexity does not mean, however, that efforts to develop useful intuitions about how to promote progress are hopeless. Because the model commonly generates phases of power-law behavior, analytic devices such as log-log plots might provide fruitful ways to compare the model's predictions to empirical data. Further, the model suggests the potential utility of dynamic-elasticity or "double ratio" tests, according to which a policy's efficacy in promoting progress is indicated by comparison of the percentage changes in push and drag parameters that the policy generates. The repeated emergence of such tests in different parameter regimes[15] suggests the possibility that the extent to which an incremental policy change speeds or slows innovative progress might commonly turn not as much on the absolute amounts by which the policy change alters one or another push or drag parameter, but more fundamentally on ratios between those absolute amounts of alteration and the parameters' pre-policy values.

14. *See* Wagner, *supra* note 8, at 1018 (positing that the amount of "open information" produced in a given time period is proportional to a weighted sum of amounts of information already in existence).

15. *See infra* text accompanying notes 118–22.

Double-ratio tests enrich intuitions about when patents or other policy instruments are most likely to promote rather than to impede progress. Indeed, these tests suggest that the effects of patent policy are even more fundamentally heterogeneous than commentators have previously appreciated. Suggestions that software and pharmaceutical patents tend to impose different magnitudes of costs on innovation might be correct,[16] and yet might also understate the heterogeneity of patents' effects. According to a double-ratio test, even if software patents increase a drag coefficient by the same absolute amount as pharmaceutical patents do, the negative effect on the software industry from that increase will be much greater if the *percentage* increase of the drag coefficient experienced by the software industry is much greater than the percentage increase experienced by the pharmaceutical industry. Hence, double-ratio tests can provide deeper insight into the often contrary views of members of different industry groups regarding the desirability of strong and broad patent rights.[17]

Part II of this Article presents the intuitive bases and mathematical structure of the model. Part II first addresses the threshold question of what is meant by "innovative progress." Part II then explains how forces promoting or impeding such progress might be expected to depend on such social factors as patent rights, the cumulative store of available knowledge, and the rate at which progress occurs.

Part III analyzes in detail the operation and implications of Part II's model for innovation dynamics. Part III shows how, according to this model, the behavior of innovative progress responds both to externally determined push and drag parameters, and also to positive and negative feedback from the speed or accumulated volume of innovative progress.

Part IV emphasizes how even Part II's relatively simple model indicates the improbability of there being a universal, natural form for the trajectory of technological progress. Further, Part IV discusses how elaborations of the model can yield the kind of *S*-shaped trajectory for technological progress that is commonly thought typical. Part IV then discusses how the repeated occurrence of such *S*-shaped behavior over finite time intervals can generate a form of "stacked tech-

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16. *See* BESSEN & MEURER, *supra* note 3, at 192 ("Clearly, software and business-method patents are different from most other patents, both in their litigation rates and frequency of claim-construction problems."); *id.* at 153 ("Although chemical and pharmaceutical firms also obtain patents covering other technologies (for example, medical instruments), the clear boundaries provided by patents on chemical structures and compositions explain the overall superior performance of the patent system in these industries.").

17. *Cf.* BURK & LEMLEY, *supra* note 5, at 3–4 (describing how representatives of the pharmaceutical industry and the information-technology industry tend to perceive "two different patent systems"); John M. Golden, *Principles for Patent Remedies*, 88 TEX. L. REV. 505, 507–08 (2010) (discussing the differing views of coalitions representing, among others, prominent pharmaceutical and information-technology companies, respectively).

nological growth" that exhibits a leading-order, long-term behavior qualitatively different from that observed at shorter time scales.

Part IV also provides an example of how log-log and semi-log plots can be used to examine time-dependent data for the sort of power-law behavior that Part II's model often predicts. In particular, Part IV shows how the cumulative number of U.S. utility patents has exhibited substantially constant power-law behavior over decades-long time scales.

Finally, Part IV shows how Part II's model yields dynamic-elasticity or double-ratio tests that can help address the question of whether a particular policy, such as an extension or strengthening of patent rights, will speed or slow innovation. Part IV explains how such double-ratio tests help both to confirm and to enrich significant intuitions about when patents are most likely to promote technological progress and when they are most likely to impede it.

II. A MODEL FOR INNOVATION DYNAMICS

A. Threshold Concerns in Constructing a Model for Innovation Dynamics

A threshold question for the construction of a model of innovation dynamics is which characteristics of those dynamics the model should attempt to describe. For purposes of this Article, it is assumed that the relevant policy goal is to increase the rate of technological progress through innovation. This goal is consistent with the common view that technological progress is generally desirable. Of particular relevance for intellectual property law in the United States, this goal is consistent with the mandate in the United States Constitution that patents operate "[t]o promote the Progress of Science and useful Arts."[18]

Undoubtedly, there are significant grounds for debate about what constitutes "technology," what constitutes "technological progress," and what the metric for any particular brand of "technological progress" should be. Economist Brian Arthur suggests that, consistent with much conventional understanding, technology can be reasonably defined as substantially human-developed "means to fulfill a human purpose"[19] that rely significantly on the understanding, operation, or manipulation of physical phenomena external to human beings.[20]

18. U.S. CONST. art. I, § 8, cl. 8.

19. W. BRIAN ARTHUR, THE NATURE OF TECHNOLOGY: WHAT IT IS AND HOW IT EVOLVES 28 (2009) (emphasis omitted).

20. *See id.* at 46 ("A technology is always based on some phenomenon or truism of nature that can be exploited and used to a purpose."); *see also id.* at 49 ("Phenomena are simply natural effects, and as such they exist independently of humans and of technology."); *cf.* John R. Thomas, *The Patenting of the Liberal Professions*, 40 B.C. L. REV. 1139, 1175

Likewise, a definition of technological progress as the development of new external, physical means to fulfill human ends seems likely to be relatively uncontroversial, at least in part because the definition seeks to be essentially objective and value-neutral.

Trickier, however, is the question of the proper metric for technological progress. Even assuming perfect knowledge of all technological means available at a particular point in time, the question of how to measure technological progress presents fundamental difficulties, in substantial part because such measurement requires settlement on an approach to assigning relative values to different technologies.[21]

On the other hand, such measurement problems are far from unique. Despite well-known problems with defining and measuring social wealth or welfare,[22] economic and legal literature is awash in analysis predicated on a goal of social-wealth or social-welfare max-

(1999) (describing technology as "knowledge that is applied toward material enterprise, guided by an orientation to the external environment and the necessity of design"). Like Arthur, I concede that an acceptable broader definition could view technology as encompassing any means to fulfill a human purpose, whether through politics, law, musical composition, reality television, or otherwise. *See* ARTHUR, *supra* note 19, at 54–56 (discussing "nontechnology-like technologies," such as "[b]usiness organizations, legal systems, monetary systems," and musical compositions, that "we can prefer to think of as purposed systems, more like first cousins to technology, even if formally they qualify as technologies"). Much of the analysis and argument that follows might apply to "technology" even when so broadly conceived. *Cf. id.* at 129 ("That origination in science or in mathematics is not fundamentally different from that in technology should not be surprising . . . . They exist because all three are purposed systems — means to purposes, broadly interpreted — and therefore must follow the same logic.").

21. *See generally* G.M. PETER SWANN, THE ECONOMICS OF INNOVATION: AN INTRODUCTION 35–36 (2009) (describing approaches to measuring innovation — such as surveys, patent counts, and investment in research and development — and their shortcomings). The economist William Baumol has acknowledged this problem of measurement even while writing extensively about innovation, saying: "[T]he very nature of the subject condemns the evidence [for proffered hypotheses] to be spotty and unsystematic, for how can one really hope to prove what has stimulated such things as invention and entrepreneurship, when neither of these can even be measured?" WILLIAM J. BAUMOL, THE FREE-MARKET INNOVATION MACHINE: ANALYZING THE GROWTH MIRACLE OF CAPITALISM 296 (2002); *cf.* Ola Olsson, *Knowledge as a Set in Idea Space: An Epistemological View on Growth*, 5 J. ECON. GROWTH 253, 253 (2000) (explaining that "thinking about knowledge and technology is complicated by the fact that ideas . . . . are practically impossible to measure and often even difficult to define").

22. *See, e.g.*, Antony Page, *Has Corporate Law Failed? Addressing Proposals for Reform*, 107 MICH. L. REV. 979, 991 (2009) ("Aggregate social welfare, including all externalities, is difficult to measure, potentially yielding widely varying estimates — and so difficult to maximize accurately."); Eric A. Posner & Adrian Vermeule, *Emergencies and Democratic Failure*, 92 VA. L. REV. 1091, 1108 (2006) ("Defining a social welfare function that aggregates across persons is notoriously difficult . . . ."); Cass R. Sunstein, *Willingness To Pay vs. Welfare*, 1 HARV. L. & POL'Y REV. 303, 330 (2007) ("I have suggested that there are serious problems with the emphasis on economic growth as the measure of social welfare . . . ."); *cf.* Richard S. Markovits, *On the Relevance of Economic Efficiency Conclusions*, 29 FLA. ST. U. L. REV. 1, 4 (2001) ("In my experience, the vast majority of economists and law-and-economics scholars . . . conflate the economic efficiency of a policy option with its social desirability when testifying before Congress, state legislatures, and administrative agencies . . . .").

imization. Presumably this at least partly reflects the fact that policymakers who would like to increase social wealth or welfare — or at least to be perceived as doing so — cannot wait for a definitive answer on how to measure such quantities. The same seems likely to be true for policymakers who seek to increase — or to be perceived as increasing — the rate of technological progress. Given the circumstances under which legal policies and decisions are made, even a relatively crude approach to measuring technological progress might help improve on mere fumbling in the dark.

At least as a theoretical matter, the measure of technological progress during a particular period of time probably should not be just the sheer number of new technological means for accomplishing human ends that have appeared during that period. Technological progress seems to require overcoming a technology-related limitation or barrier, whether through a flash of insight, a substantial targeted investment, or an effort to develop a market that will support a new technology's diffusion and improvement. Many new technological means might fail to qualify as progress because, for example, they are virtually identical to others or easily developed from them. Thus a straightforward count of the number of new product lines, methods of manufacturing, or other technological means might entail substantial over-counting of instances of true technological progress.

Further, some new technological means represent more significant steps forward than others. United States patent law recognizes this fact through a requirement that a patentable invention not "have been obvious at the time the invention was made to a person having ordinary skill in the [relevant] art."[23] More subtly, U.S. patent law incorporates an effective judgment about such significance by generally restricting the prior art relevant for assessing novelty and nonobviousness to art that, by one or another specified point in time, was publicly known, publicly disclosed, or commercially exploited,[24] or at least put on a path reasonably likely to yield such public knowledge, public disclosure, or commercial exploitation.[25] Generally speaking,

23. 35 U.S.C. § 103(a) (2006).

24. *See* 35 U.S.C. § 102(a) (2006) (defining prior art that can render an alleged invention anticipated as including matter "known or used by others in this country, or patented or described in a printed publication in this or a foreign country, before the invention thereof by the applicant for patent"); *id.* § 102(b) (treating as prior art matter "patented or described in a printed publication in this or a foreign country or in public use or on sale in this country, more than one year prior to the date of the application for patent in the United States"); MERGES & DUFFY, *supra* note 11, at 396 (observing the accepted U.S. understanding that "prior knowledge or use must be *public* in order to qualify as anticipatory"); *id.* at 730 ("There has never been any doubt that . . . prior art [for the purpose of assessing nonobviousness] includes references under § 102(a), all of which are . . . publicly available.").

25. *See* 35 U.S.C. § 102(e) (treating matter in an issued U.S. patent or published U.S. patent application as prior art as of the date of U.S. filing); *id.* § 102(g)(2) (treating as prior art matter "made in [the U.S.]" and "not abandoned, suppressed, or concealed"); Flex-Rest, LLC v. Steelcase, Inc., 455 F.3d 1351, 1358 (Fed. Cir. 2006) ("There are two types of sup-

the patent laws of other countries contain similar provisions.[26] Hence, patent law generally reflects a judgment that the technological advances that matter most for purposes of measuring progress are those whose structure, operation, or benefits are made publicly available in some significant way.[27]

Settlement on a meaningful quantitative measure of technological progress is likely to be even more difficult than the binary judgments of public availability and patentability that patent law tends to make. In general, one will likely have to use a measure that is at best a crude proxy for technological progress itself.[28] For example, the number of patentable innovations generated during a given time period — whether ultimately patented or not — might be viewed as one measure of technological progress. But such a measure would be less than ideal.[29] Some important innovations might not meet the threshold of patentability — for example, because they fall outside patent law's subject-matter boundaries, or because they involve non-novel or obvious variants of pre-existing products or processes that failed to achieve commercial success.[30] Moreover, because patentable innova-

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pression or concealment: cases in which the inventor intentionally suppresses or conceals his invention, and cases in which a legal inference of suppression or concealment can be drawn based on an unreasonable delay in making the invention publicly known.").

26. *See, e.g.*, GRAEME B. DINWOODIE, WILLIAM O. HENNESSEY & SHIRA PERLMUTTER, INTERNATIONAL AND COMPARATIVE PATENT LAW 120–21 (2002) (discussing definitions of prior art in Europe and Japan).

27. Some economic models suggest that some degree of spillover-enabling public disclosure might be crucial for sustaining economic growth. *See* GENE M. GROSSMAN & ELHANAN HELPMAN, INNOVATION AND GROWTH IN THE GLOBAL ECONOMY 57 (1991) (contending that, in accordance with certain models, full appropriation of research-and-development returns "by the inventor" results in "the cessation of growth in the long run").

28. *Cf.* ÅKE E. ANDERSSON & MARTIN J. BECKMANN, ECONOMICS OF KNOWLEDGE: THEORY, MODELS AND MEASUREMENTS 56–57 (2009) (asserting that, in measuring "knowledge inputs," "[t]he best one can do is using proxy variables such as the literature on a subject"); Christian Ghiglino, *Balanced Growth with a Network of Ideas* 2 (Dep't of Econ., Queen Mary Univ. of London, Working Paper No. 546, 2005), *available at* http://ideas.repec.org/p/qmw/qmwecw/wp546.html ("[D]ata about the production of 'ideas' are impossible to obtain. The most closely related processes are the production of *patents* and the production of *scientific articles*.").

29. Presumably even less ideal would be a measure of progress based on raw numbers of issued patents, which are unlikely to reflect all innovations that, in principle, could have been patented, as various important innovations might have been either held as trade secrets or made freely available. Nonetheless, patent counts are commonly used as a crude measure of technological progress. *See, e.g.*, *Climbing Mount Publishable*, THE ECONOMIST, November 13th–19th 2010, at 95, 96 (using patent counts as a measure of nations' "success[] in using the knowledge they generate").

30. *See* ROGER CULLIS, PATENTS, INVENTIONS AND THE DYNAMICS OF INNOVATION: A MULTIDISCIPLINARY STUDY 6 (2007) (noting that "technical advances may fail to satisfy statutory tests for quantum of inventive step"). U.S. patent law considers the commercial success of a product or process incorporating a claimed invention to be potential evidence of the claimed invention's nonobviousness, but that law also recognizes that such commercial success can result from factors having little or nothing to do with the technological merit of the claimed invention. *See, e.g.*, Ormco Corp. v. Align Tech., Inc., 463 F.3d 1299, 1311–12 (Fed. Cir. 2006) ("Evidence of commercial success, or other secondary considera-

tions can differ greatly in economic or technological significance, a simple count of patentable innovations can misrepresent the importance even of the innovations counted.[31] On the other hand, if the probability distribution of such innovations' value or significance is relatively stable over time — i.e., if the percentage of innovations in a given time period that have a particular value or significance is essentially constant — counts of patentable innovations might provide a reasonably good measure of technological progress.

Alternatively, if one is concerned with progress in a specific technological field such as semiconductor chip production, one might define technological progress according to technical benchmarks, such as the number of transistors per unit area or the microprocessor speed achievable per dollar of production cost. But even if concern with progress is limited to a specific technological field, such a technical metric for progress is likely to be problematic. Progress according to a technical metric does not necessarily bear any substantial correlation to the difficulty or social significance of the advance. Thus, however useful such metrics might be as a means of setting engineering goals within a firm, their use for purposes of setting social policy might arbitrarily shape perceptions of the state of technological progress and the dynamics behind it. For example, using the number of transistors on a semiconductor chip as a measure of progress would indicate that progress has proceeded exponentially with time.[32] On the other hand, using a logarithm of the number of transistors per unit area as the measure causes progress to look substantially linear.[33] Moreover, any such metrics fail to shed much light on questions likely to be of much greater concern — such as how much effort or ingenuity was required to increase the density of transistors, or how much this achievement has improved workplace productivity, working or living conditions, or social capacity to address any of a number of human wants.

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tions, is only significant if there is a nexus between the claimed invention and the commercial success . . . . Thus, if the commercial success is due to an unclaimed feature of the device, the commercial success is irrelevant.").

31. *Cf.* CULLIS, *supra* note 30, at 11 (observing that patent office statistics do not "distinguish between ideas which make a significant contribution to scientific and economic progress and those which are mere paper proposals"); SWANN, *supra* note 21, at 35–36 ("[I]t is generally reckoned that most patents cover inventions of pretty low commercial value . . . . In short, the existence of patents does not necessarily imply innovation and the absence of patents does not necessarily imply the absence of innovation.").

32. *See* Robert Service, *Intel's Breakthrough: Its New Silicon Laser Could Add Decades to Moore's Law*, 108 TECH. REV. 62, 62 (2005) (discussing efforts to extend the period of at least approximate validity for "Moore's Law," a decades-old prediction "that the number of transistors on a computer chip would double every two years").

33. *See* Intel, *Moore's Law: Made Real by Intel Innovations*, http://www.intel.com/technology/mooreslaw/ (last visited Dec. 21, 2010) (showing how Moore's Law of exponential growth becomes a law of linear progress when a logarithm of transistor size is plotted as a function of time).

With such concerns in view, one might assign a numerical value to an innovation that reflects what it generates: whether citations in relevant patent or academic literature, or some at least theoretically measurable economic surplus — perhaps ideally the net social surplus expected to result from the innovation.

Given the pragmatic concerns of many U.S. legislators and citizens, such an economically oriented approach to assigning value to innovation might seem desirable for purposes of developing U.S. innovation policy. But an effort at accurate economic assessment might raise intractable questions. For example, should the measure for innovation reflect the extent to which the innovation's value accrues to people in the United States alone? If an innovation saves lives, what are these savings worth?[34] To what degree should we discount the economic value of an innovation that goes unrealized for decades?[35] Recognizing that the opportunity costs of an innovation's development and dissemination might involve foregone or delayed progress with respect to other innovations, entrepreneurial activity, or alternative social developments of uncertain value,[36] how should we calculate the total costs of an innovation? Moreover, how does one attribute economic value to particular innovations, given their frequently cumulative or complementary nature? For example, if both new software and a more efficient design of internal circuitry are necessary for the broad adoption of a new smartphone, and if neither that software nor that circuit design have significant uses outside of the smartphone, how does one determine how much of the smartphone's value should be assigned to the new software and how much should be assigned to the new circuitry?[37] For that matter, how much of the smartphone's value should be attributed to prior innovations — such as commercializable cellphone technology, the transistor, and Maxwell's laws of electromagnetism — whose great and lasting value has been further proven by the subsequent innovations that led more proximately to the smartphone?

In short, settling on a satisfactory measure of technological progress is difficult. Indeed, the nature of the best relevant metric seems

34. *Cf.* Frank Ackerman & Lisa Heinzerling, *Pricing the Priceless: Cost-Benefit Analysis of Environmental Protection*, 150 U. PA. L. REV. 1553, 1564 (2002) ("What can it mean to say that saving a life is worth $6.3 million?").

35. *Cf.* Cass R. Sunstein, *Cost-Benefit Default Principles*, 99 MICH. L. REV. 1651, 1711 (2001) ("Perhaps the most difficult issue here, from the theoretical point of view, involves the selection of the appropriate discount rate.").

36. *Cf.* BAUMOL, *supra* note 21, at viii ("[W]hen institutional arrangements happen to offer greater rewards to enterprising rent-seeking or to destructive activities such as warfare or organized crime than they offer to productive entrepreneurial activity, we can expect an economy's entrepreneurial effort to be allocated away from the more productive undertakings.").

37. *See* Golden, *supra* note 17, at 535–36 (discussing potential difficulties in assigning value to different parts of a multi-component technology).

likely to depend on the policy question at hand. The goals of ensuring the long-term survival of *Homo sapiens* or its possible descendants, the long-term continuance of the United States as an economic and military superpower, or the short-term prosperity of the United States' voting public might lead to different preferred metrics for technological progress. Nonetheless, the U.S. Constitution, by making an interest in promoting progress the stated basis for enacting patent and copyright laws,[38] implicitly presumes that policymakers can make rational judgments about what progress is and whether the current rate of progress is satisfactory. The model presented in Part II.B likewise presumes the existence of a rational metric for technological progress. Although the model does not require that any particular metric be used, it does reflect the previously discussed intuition that technological progress involves overcoming a technology-related limitation or barrier.[39]

B. The Basic Model and Its Explanation

Assuming that we have settled on a measure of technological progress, what would we expect to accelerate or to decelerate such progress? Economic analysis of whether innovation is likely to occur typically considers whether the cost of an innovative step is likely to be outweighed by the benefit to one or more actors of completing that step. According to this view, innovation operates as a process that is fostered by incentives and slowed by costs.

Building on this conventional view, this Article uses a model for innovative progress in which the rate of progress is determined by a combination of accelerant "pushes" and decelerant "drags." Pushes can reflect increased or increasing demand for innovation, investment in innovation, or policies promoting such investment or demand. Drags can reflect any of a number of potential obstacles to the development or deployment of innovation, such as the difficulty of solving a technological problem or a regulatory hurdle to distributing or using an innovation. A particular policy tool such as patent rights might contribute nonnegligibly to both pushes and drags.

Consistent with the above, the model for innovation dynamics used here corresponds to a Newtonian equation of motion for a body that is propelled forward by pushes, but that also encounters resistance in the form of velocity-independent and velocity-dependent drags. The basic mathematical formulation for the model used here is given by Equation 1 below:

$$dv/dt = \alpha + \beta x^{\varepsilon} - \mu - \gamma v - \xi v^{1+\eta} \qquad \text{(Eq. 1).}$$

38. U.S. CONST. art. I, § 8, cl. 8.

39. *See supra* text accompanying notes 22–23.

The elements of this equation can be described as follows:

- The quantity $x(t)$ is a function of time that represents the cumulative amount of technological progress that has been achieved by time t, where such an amount is assumed to be nonnegative.
- The quantity $v(t)$ is a function of time, for purposes of simplicity also assumed to be nonnegative, that represents the speed of technological progress, dx/dt, at time t.
- The quantity dv/dt is a function of time that represents the acceleration of progress — i.e., the rate at which the speed of progress is changing at a particular point in time.
- The Greek letters ε and η represent dimensionless real numbers that are assumed here to be positive and constant in time.[40]
- The letters α, β, κ, γ, and ξ are assumed to represent real values of appropriate dimensionality (i.e., units of technological progress divided by units of time squared for α) that are both nonnegative and, absent a policy change, constant in time.
- The letter α represents the magnitude of the total push forward that is not captured by the subsequent positive-feedback term.
- The quantity βx^{ε} is a positive feedback term having $0 < \varepsilon \leq 1$ and representing, to leading order in x, the extent to which the cumulative magnitude of past innovation drives further innovation.
- The letter μ represents the magnitude of a drag on technological progress that is analogous to a term of kinetic or static friction.[41]
- The quantity γv captures the extent to which drags on innovation increase linearly with the speed v at which innovation occurs.

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40. The assumption of constant ε and η might be viewed as controversial. The assumption is made here primarily for purposes of simplicity. *Cf.* ROBERT M. SOLOW, GROWTH THEORY: AN EXPOSITION 98, 181 (2d ed. 2000) (observing that models of economic growth have commonly treated parameters such as the rate of technological progress or population growth as exogenous, and arguing that such treatment is a proper "temporary designation" when there is insufficient understanding of how to incorporate such parameters in "a determinate cause-and-effect model"). On the other hand, the time-dependence of the cumulative number of U.S. utility patents, discussed in Part IV, *infra*, suggests that an assumption that the exponents ε and η are constant over significant time periods might be reasonable. Over three decades-long time periods from 1790 through 2009, the cumulative number of U.S. utility patents has grown fairly consistently in accordance with constant-exponent power laws. *See infra* Figures 15–17 and accompanying text.

41. *See* DOUGLAS C. GIANCOLI, PHYSICS: PRINCIPLES WITH APPLICATIONS 83–85 (3d ed. 1991) (discussing kinetic and static friction). Like a force of static friction, μ takes the lower of the two values μ_s and $\alpha + \beta x^{\varepsilon}$ when the speed of progress v equals zero. *See id.* at 84–85. The term μ might have a different positive value μ_k when v is greater than zero. *Cf. id.* at 84.

- The quantity $\xi v^{1+\eta}$ captures the leading supralinear dependence of drags on the speed v.

Significantly, the intuitions behind Equation 1's two innovation-accelerating terms, α and βx^{ε}, and Equation 1's three innovation-decelerating terms, $-\mu$, $-\gamma v$, and $-\xi v^{1+\eta}$, are relatively insensitive to the precise manner in which innovative progress is defined. As long as such progress tends to encounter regular resistance and responds to accelerants, the general intuitions behind these terms appear plausible. Discussion of these intuitions follows.

The basic acceleration term α represents an x-independent push derived from the combined influence of any of a number of levers — e.g., investment in research and development, government provision of patent rights, or prizes and reputational benefits for discoverers and developers — that could either speed innovative progress or offset the decelerating effect of Equation 1's drag terms. For example, providing a monetary reward for a particular technological advance is likely to accelerate innovation, whether by increasing the number of participants in a race to make that advance, or by increasing the investments of participants in such a race. Increased participation or investment in innovation will likely result in both faster and more certain success in the race.[42]

The second acceleration term βx^{ε} demands more explanation. As indicated above, this term means that cumulative know-how and technological capacity has a positive feedback into the process of innovation itself, leading to an acceleration of innovation as society's overall store of technology grows. Such feedback can occur in multiple ways. First, technologies can generate enhanced economic growth that in turn supports or encourages: (1) increased direct investment in technological progress,[43] (2) increased investment in the education of individuals who will be well-placed to use new technologies and to improve on them,[44] and (3) increased wealth that can support broad-based and well-financed demand for new technologies.[45] Second, ex-

42. *See* SCOTCHMER, *supra* note 2, at 112–13 (describing how "patent races can accelerate progress" by increasing participation or investment in innovation).

43. *See* BAUMOL, *supra* note 21, at 286 (observing that "[s]uccessful innovation . . . widens the acceptance of innovative products by business enterprise," "makes entry into the activity [of innovation] more attractive to others, and makes it easier to raise the requisite capital").

44. *Cf. id.* at 13 ("Only the productive surpluses that innovation began to make possible, first in agriculture and mining and then in manufacturing, made feasible the enormous increases in investment in inanimate and in human capital that are widely judged to have contributed greatly to economic growth.").

45. *See* ARTHUR, *supra* note 19, at 175 ("[B]ecause societies prosper as their technologies build out, our needs grow as technology builds out."); *cf.* CULLIS, *supra* note 30, at 67 ("As a result of the incandescent lamp people were active for a greater proportion of the day . . . . Clearly, a major effect of innovation is an overall growth in consumption."); Michael Kremer, *Population Growth and Technological Change: One Million B.C. to 1990*, 108 Q.J. ECON. 681, 681–82 (1993) (developing "a highly stylized model" in which "the

isting technologies can contribute to positive feedback by generating demand for new technologies to supplement, service, or otherwise improve existing technologies.[46] For example, the development and popularization of the automobile has helped generate demand and support for any of a number of successive technological developments relating to aerodynamic design, passenger safety, automated windshield wipers, bridge and highway construction, portable radar and radar detection, oil detection and extraction, and polymer design and manufacturing for the construction of tires.[47] Finally, the magnitude of cumulative technological know-how can also help speed and even inspire further technological progress by providing a store of technological elements, tools, or techniques that are ready-made for use in the development of new patentable combinations.[48] As society's tech-

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growth rate of technology is proportional to total population" and "the growth rate of population is proportional to the growth rate of technology").

46. *See* ARTHUR, *supra* note 19, at 175 (observing that "every technology requires supporting technologies"); BAUMOL, *supra* note 21, at 285 (noting the phenomena of "one invention call[ing] for another to make it more effectively workable," and of a "new product . . . invit[ing] R&D devoted to [the product's] improvement . . . or to the creation of superior substitutes . . . or even to improvement of an old product that is threatened with replacement").

47. *Cf.* ARTHUR, *supra* note 19, at 175–76 (contending that "[t]he vast majority of niches for technology are created not from human needs, but from the needs of technologies themselves," and that "[t]echnologies often cause problems — indirectly — and this generates needs, or opportunities, for solutions").

48. *See* ARTHUR, *supra* note 19, at 21 ("These new technologies in time become possible components — building blocks — for the construction of further new technologies."); BAUMOL, *supra* note 21, at 12 (observing that "one invention may . . . indicate ways to make it easier and less costly to manufacture other new products," and that "the innovation process itself leads to improvements in the way R&D is carried out"); GROSSMAN & HELPMAN, *supra* note 27, at 17 ("[I]nnovation conceivably can be a self-perpetuating process. Resources and knowledge may be combined to produce new knowledge, some of which then spills over to the research community, and thereby facilitates the creation of still more knowledge."); Paul M. Romer, *Endogenous Technological Change*, 98 J. POL. ECON. S71, S83 (1990) ("[T]he larger the total stock of designs and knowledge is, the higher the productivity of an engineer working in the research sector will be."); Martin L. Weitzman, *Recombinant Growth*, 113 Q.J. ECON. 331, 336 (1998) (developing a model for innovation based on a "combinatoric metaphor"). In reality, positive feedback from existing technology's utility as a toolkit or set of building blocks for further advance might be understood to operate in two different ways: (1) through existing technology effectively providing ideas and inspiration for new technological developments — e.g., the separate successes of handheld wireless phones and of the Internet suggesting the desirability of providing Internet access through a wireless handheld — a "combinatorial mechanism" that would seem relatively naturally modeled by a positive-feedback term such as βx^{ε}, *cf.* ARTHUR, *supra* note 19, at 20 ("[T]he very cumulation of earlier technologies begets further cumulation."); or (2) through existing technology making the process steps for developing a distinct new invention easier — e.g., graphical-design software lowering the costs of product design by reducing the need for handcrafted models — a "research-tool mechanism" that would seem more naturally modeled by a reduction in drag coefficients such as γ, *see* GROSSMAN & HELPMAN, *supra* note 27, at 58 (describing a model in which technological advances "reduce the labor requirements for designing new products"); Charles I. Jones, *R & D-Based Models of Economic Growth*, 103 J. POL. ECON. 759, 765 (1995) ("The discovery of calculus, the invention of the transistor, and the creation of semiconductors are all examples of major innovations that most likely raised the productivity of the scientists who followed.").

nological toolkit grows, a developmental optimist might follow suggestions from sociologist William Ogburn,[49] economist Brian Arthur,[50] and legal scholar Polk Wagner[51] and expect the rate of technological progress to accelerate ever more rapidly.

On the other hand, Equation 1 departs from models developed by Polk Wagner and others by allowing the positive-feedback exponent ε to take on values other than one. Wagner's model for information production assumes such linear feedback and, by further assuming the absence of supralinear negative feedback, predictably generates exponential growth.[52] Likewise, Brian Arthur[53] and more effusive futurists such as Raymond Kurzweil[54] have explicitly argued that exponential growth is a natural consequence of such feedback.[55] Such positive-feedback optimists might find support in the recent explosive growth

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For purposes of simplicity, Equation 1 only models positive feedback in the manner of the first kind of mechanism. The second mechanism seems limited in the forms of behavior it can produce because the most that it can do is reduce the size of drag coefficients. Thus, the text focuses on the first mechanism, viewing this form of feedback as a potential contributor to the term βx^{ε}.

49. *See* WILLIAM FIELDING OGBURN, SOCIAL CHANGE WITH RESPECT TO CULTURE AND ORIGINAL NATURE 104 (Peter Smith ed., 1964) (1922) ("It would seem that the larger the equipment of material culture the greater the number of inventions.").

50. *See* ARTHUR, *supra* note 19, at 173–74 (discussing Ogburn's speculation about exponential growth, and observing that "crude combinatorics demonstrate that if new technologies lead to further new technologies, then once the numbers of elements in the collective pass some rough threshold, the possibilities of combination begin to explode").

51. *See* Wagner, *supra* note 8, at 1018, 1021 (positing that the amount of "open information" produced in a given time period is proportional to a weighted sum of amounts of information already in existence).

52. *See id.* at 1018–22.

53. *See* ARTHUR, *supra* note 19, at 173 (providing "theoretical reasoning to back [Ogburn's] exponential claim"). Although Arthur attributes to Ogburn a theory of exponential growth of technology, Ogburn in fact presents a more qualified model of exponential growth, for he explicitly recognizes the likelihood of progress-damping terms that will keep the sum total of what Ogburn calls "material culture" from growing exponentially for a very extended period of time. *See* OGBURN, *supra* note 49, at 113 ("In fact it is difficult to conceive of any growth under actual conditions increasing for long according to the compound interest law . . . . Malthus said that population tended to increase in a geometric progression; but it is only a tendency for there are actual checks.").

54. *See* Ray Kurzweil, *The Law of Accelerating Returns*, KURZWEIL ACCELERATING INTELLIGENCE (Mar. 7, 2001) [hereinafter Kurzweil, *Law*], http://www.kurzweilai.net/the-law-of-accelerating-returns ("Exponential growth is a feature of any evolutionary process, of which technology is a primary example."); *cf.* Ray Kurzweil, *Making the World a Billion Times Better*, WASH. POST, Apr. 13, 2008, at B4 (appearing more generally to restrict the claim of exponential growth to "information technology," understood broadly to include nanotechnology and modern "biological technologies").

55. *Cf.* Graham Lawton, *The Incredibles*, NEW SCIENTIST, May 13, 2006, at 32, 35 (2006) ("[M]ost technologies are advancing not linearly, but exponentially."). *But cf.* Javier Carrillo-Hermosilla & Gregory C. Unruh, *Technology Stability and Change: An Integrated Evolutionary Approach*, 40 J. ECON. ISSUES 707, 727 (2006) (observing that, while positive feedback generating "[constant change and innovation] seems to accurately define technologically dynamic sectors like computers and information technology, other industries are far more quiescent").

of "digital information" generated each year[56] or the semiconductor industry's decades-long success in staying on track with Moore's law for the development of increasingly complex computer chips.[57]

There seems good reason to suspect, however, that — at least as the cumulative store of innovation becomes very large — positive-feedback effects will typically be less than linear, and technology will periodically run into barriers or bottlenecks that cause a dramatic slowing of growth.[58] As the store of cumulative technologies and knowledge grows, the spectrum of technologies and know-how might become more diverse, and the proportion of technologies or know-how relevant to the development of any particular new technology might thus become smaller. Further, as the store of technologies and know-how grows, the chance that many past technologies will only make substantially redundant contributions to the development of a new technology is likely also to grow. Moreover, although technological progress and the economic growth that it commonly promotes can increase demand for newer technologies, such demand might ultimately become saturated with respect to a particular technology or set of technologies.

Thus, for a variety of reasons and particularly when the development of a specific set of technologies is at issue, a phenomenon of diminishing returns with respect to positive technological feedback seems likely.[59] At least for very large values of x, there seems a good chance that the exponent ε of the term βx^{ε} will be less than 1. As discussed in Part III, when this occurs the positive-feedback term βx^{ε} tends to support power-law growth in the manner of t^{z}, rather than exponential growth in the manner of $e^{\lambda t}$, where t represents time and the quantities z and λ are positive real numbers.

The first of the innovation-decelerating terms, $-\mu$, is essentially the "frictional" counterpart of the innovation-accelerating term α. This frictional term might be viewed as reflecting overhead costs of innovative progress that are independent of the speed of progress, v, as long as that speed is greater than 0.

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56. Kenneth Cukier, *Data, Data Everywhere*, THE ECONOMIST (SPECIAL REP. ON MANAGING INFO.), February 27th–March 5th 2010, at 1, 2 ("The amount of digital information increases tenfold every five years.").

57. *See supra* notes 32–33 and accompanying text.

58. *Cf.* JOEL MOKYR, THE LEVER OF RICHES: TECHNOLOGICAL CREATIVITY AND ECONOMIC PROGRESS 292 (1990) ("[W]ithout macroinventions, microinventions are likely eventually to run into diminishing returns, and technology will begin to look more and more [to be in a state of] 'stasis' . . . unless and until a new major invention occurs."); William J. Baumol & Edward N. Wolff, *Feedback from Productivity Growth to R & D*, 85 SCANDINAVIAN J. ECON. 147, 149–50 (1983) (observing that productivity growth in a particular area will likely become "asymptotically stagnant" as costs become dominated by factors for "which productivity growth is inherently negligible," such as many forms of human services (emphasis omitted)).

59. *See* Karjala, *supra* note 8, at 13 ("We also know from nearly every sphere of activity that there is a law of diminishing returns.").

The second deceleration term, $-\gamma v$, represents first-order effects of speed-dependent impediments to innovative progress. The intuition behind this term can be explained in at least two ways. First, one might view innovative progress as a process of overcoming successive hurdles, which might include technological or legal obstacles,[60] resource or infrastructure constraints,[61] and public resistance.[62] Each hurdle encountered would, in the absence of countervailing innovation-accelerating forces, cause the rate of innovation to slow down by the amount ψ. On average, such hurdles are spaced at intervals of width δ along the path of innovative progress. Consequently, the number N of hurdles encountered in an infinitesimal time interval Δt during which progress proceeds with average speed v is given by the distance traveled during that time interval, $v\Delta t$, divided by the hurdle interval δ; in short, $N = v\Delta t/\delta$. The net slowdown caused by the hurdles during the interval Δt is $N\psi$ or $(\psi/\delta)v\Delta t$. The net slowdown per unit time is therefore $(\psi/\delta)v$. Setting γ equal to the ratio ψ/δ yields the deceleration term $-\gamma v$.[63]

A less mathematical explanation for the intuition behind the deceleration term $-\gamma v$ might go as follows. Suppose that an actor wants to develop an innovation meeting certain functional specifications in

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60. *See* CULLIS, *supra* note 30, at 33 ("Innovation is, at times, inhibited by the need for some enabling invention or other event, such as the development of a material having suitable physical characteristics or the enactment of legislation which removes regulatory barriers.").

61. *See id.* at 69 ("The more infrastructure creation that is necessary, the slower will be the introduction of an innovation . . . ."); *id.* at 196 ("[T]he greater the change which is brought about by the invention, the lower will be the probability that the wherewithal to bring it into effect will be in existence.").

62. *Id.* at 25 (discussing "the difficulty of communication and gaining acceptance of new ideas").

63. One might imagine situations in which the linear-drag coefficient γ grows with x, the amount of technological progress already achieved. Many commentators have theorized that "technological improvements become increasingly difficult as the threshold for new discoveries rises," a phenomenon that could explain an apparent long-term trend of increased "research effort" per patent obtained. Samuel S. Kortum, *Research, Patenting, and Technological Change*, 65 ECONOMETRICA 1389, 1392 (1997); *see also* Jones, *supra* note 48, at 765 ("[P]erhaps the most obvious ideas are discovered first so that the probability that a person engaged in R & D discovers a new idea is decreasing in the level of knowledge."). On the other hand, recent rapid growth in patenting might call this trend and its explanation into question. *See supra* note 9 and accompanying text; *cf.* BESSEN & MEURER, *supra* note 3, at 179 fig.8.2 (showing that, following a rise in U.S. patents granted to U.S. inventors starting around 1980, the ratio of the number of domestic grants to the number of U.S. residents has risen to a level that roughly matches peak levels of the 1940s); John M. Golden, *"Patent Trolls" and Patent Remedies*, 85 TEX. L. REV. 2111, 2111 n.3 (2007) (observing that the average number of issued U.S. patents per billion dollars of real gross domestic product was at about the same level from 2000 to 2005 as it was during the 1970s). Given uncertainty in this regard, and given that the present model already generates a complex array of potential behaviors, this further potential negative-feedback effect is not central to the discussion and application of Equation 1 in Parts II and III, and is left for further discussion in Part IV.A.

six months, rather than one year.[64] The actor previously calculated that it would require an investment of $\$X$ worth of capital and labor to develop this innovation over the course of one year. The actor might reasonably expect that, to produce the innovation in half that time, it will need to spend at least the same amount $\$X$ over the course of six months. In other words, the actor might expect that doubling the rate of innovative progress requires at least doubling the rate of investment, absent any efficiency-enhancing synergies from compressing the timeline for innovation. The deceleration term $-\gamma v$ similarly suggests that an effort to increase the rate of innovative progress will require at least a corresponding linear increase in forces that propel innovation forward.

The supralinear deceleration term $-\xi v^{1+\eta}$ embodies the intuition behind the qualifier "at least" in the statement that doubling the rate of innovative progress will require at least doubling the rate of investment. The law of diminishing returns is likely to apply here as elsewhere.[65] Capital for increased investment might become harder to acquire as further increments are demanded. As the workloads of current employees increase, they might become less productive, either per unit time or per dollar spent. Ramping up efforts by hiring new employees of a competence level equal to that of current employees might also become more difficult with each additional increment of hirees required. The list of potential causes of a supralinear increase in the forces opposing an increase of the rate of innovative progress could go on and on. Consequently, the basis for suspecting the existence of higher-order deceleration terms like $-\xi v^{1+\eta}$, where η is greater than zero, seems robust.

In short, common intuitions about potential causes for the acceleration or deceleration of technological progress support including the five terms on the right side of Equation 1. Other terms might be added, and, as Part IV demonstrates with respect to the drag parameter γ, Equation 1's Greek-lettered parameters — α, β, γ, etc. — could themselves be represented as having some combination of t-dependence, x-dependence, and v-dependence. On the other hand, for an initial effort at describing the dynamics of a process involving advance in the face of obstacles, Equation 1 likely incorporates sufficient complexity.[66] Even Equation 1's bounded degree of complexity reveals severe limi-

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64. *Cf.* Gillette Co. v. Energizer Holdings, Inc., 405 F.3d 1367, 1369 (Fed. Cir. 2005) (noting that development of a successful multi-blade razor had required "overcom[ing] the undesired drag forces produced by razors with multiple blades").

65. *See* Paul M. Romer, *Increasing Returns and Long-Run Growth*, 94 J. POL. ECON. 1002, 1020 (1986) ("That the production of new knowledge exhibits some form of diminishing marginal productivity at any point in time should not be controversial.").

66. The intuitions supporting Equation 1 as a model for innovation dynamics seem likely to extend to dynamic processes in which obstacles to continued progress are regularly encountered.

tations to arguments that technological growth naturally proceeds exponentially. Additionally, this degree of complexity suggests double-ratio tests for evaluating policy proposals that do not appear to have been recognized in prior work.[67]

Before proceeding to a more detailed account of the implications of Equation 1 for innovation dynamics, I should add some caveats regarding the degree of "technology optimism" that the model reflects. As noted above, the model assumes that the rate of innovative progress v is never negative. In accordance with this constraint on the model, the model never predicts regress, only stasis or progress. Neither minor regresses — such as ones due to occasional social forgetfulness — nor major regresses — whether from epidemic, war, natural disaster or otherwise — are modeled here.

Similarly, any feedback from accumulated technological progress is assumed to be positive. This need not be the case. For example, modern technology has enabled the production of weapons of mass destruction that have increased the likelihood that warfare will snuff out technological progress or human civilization itself. More generally, success in developing human capacities to shape and control our environment might carry with it an increased risk that, through an error in judgment or a failure to exercise judgment, humanity might undercut environmental conditions that support both itself and technological progress.[68] But like less human-based catastrophes, such dystopian possibilities for negative feedback are not modeled here.

A further, disputably optimistic aspect of the model is more subtle. As must be expected of a model for acceleration or deceleration that tracks Newtonian models for the motion of physical bodies, the model implicitly incorporates a notion of inertia: a sense that, if all relevant forces are removed — e.g., if all the terms on the right-hand-side of Equation 1 are set to zero — progress will proceed at a constant, possibly nonzero rate. I must confess that my own intuitions naturally resist this notion that progress could effortlessly persist forever, with progress continuing indefinitely at the rate at which it happened to be occurring when all forces were turned off. But such intuitions might substantially be driven by the ubiquity, in practical experience, of motion-damping terms such as the drag terms of Equation 1.

Given the current crude state of our understanding of innovation dynamics and the general failure, so far, to examine how to describe those dynamics via a differential equation of motion, it seems enough

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67. *See infra* Part IV.C.

68. *Cf.* OGBURN, *supra* note 49, at 117 ("It is probable that the social development of the future will be affected by changes in the quantity and nature of natural resources such as soil, minerals, forests, etc. . . . So changes, shortages, or discoveries in natural resources will have an effect on the material culture of the future.").

for now to include drag terms that appear likely to bring the operation of the equation's dynamics within the range of our common intuitions and experience, and then to see where the model leads us.[69]

III. Operation of the Model

Perhaps the best way to understand the operation of the model is to investigate how the speed of innovation behaves in different regimes for the model's input parameters. To this end, analysis of the model's operation is divided into two basic parts: (A) investigation of "small-v dynamics" for circumstances where the supralinear drag term $-\xi v^{1+\eta}$ can be treated as negligible compared to the linear drag term $-\gamma v$, and (B) investigation of "large-v dynamics" for circumstances where this is not the case. Within each of these parts, analysis proceeds via subparts looking at situations (1) where the positive-feedback term βx^{ε} can be treated as negligible or, at least, as negligibly variable and thus absorbable into the acceleration term α, and (2) where the positive-feedback term βx^{ε} is large and its time-dependence is significant for behavior during the time period studied.

A. Small-v Dynamics

The first set of parametric regimes to be considered are those for which the nonlinear drag term $-\xi v^{1+\eta}$ can be treated as negligible. To a first approximation, this seems a reasonable assumption if, throughout the time period of interest, the magnitude of the nonlinear drag term $-\xi v^{1+\eta}$ is generally much less than the magnitude of the linear drag term $-\gamma v$ — i.e., if $\xi v^{1+\eta} << \gamma v$. This inequality holds if (1) $\xi = 0$ and v and γ are both positive or (2) if $v << (\gamma/\xi)^{1/\eta}$. Because of the latter inequality, this regime is called a "small-v" regime although, for ξ equal to zero or much smaller than γ, the speed of innovative progress v need not be small in a more general sense of the term.

69. Alternatively, one might try to avoid the problem of a counterintuitive notion of an inertial state of innovative progress by adopting a substantially different vision of innovation dynamics. For example, one might adopt an "Aristotelian mechanics" according to which the rate of progress is simply proportional to the ratio between forces of propulsion and forces of resistance to motion. *Cf.* GERALD HOLTON & STEPHEN G. BRUSH, INTRODUCTION TO CONCEPTS AND THEORIES IN PHYSICAL SCIENCE 80–81 (2d ed. 1985) (describing the Aristotelian account of motion). But such a model leads to problems that are perhaps even more fundamental, such as the lack of an account of acceleration, and the question of whether such Aristotelian proportionality should hold when the forces of resistance exceed the forces of propulsion. *Cf. id.* at 81 (discussing how the Aristotelian account of motion was internally inconsistent).

1. Terminal Velocity for Negligibly-Variable Positive Feedback

Equation 1 resolves to a simpler formula when the positive-feedback term βx^{ε} has negligible time dependence. Of course, this is true when $\beta = 0$ — i.e., when there is no positive feedback — but, because the existence of some positive feedback seems plausible under many circumstances, we might like to consider more general conditions under which the time dependence of βx^{ε} is negligible.

To a first approximation, we can ignore the term βx^{ε} if it is much less than the minimum magnitude of $\alpha - \mu$ throughout the time period of interest — i.e., if $(\beta x^{\varepsilon})_{\max} << |\alpha - \mu|_{\min}$ throughout that period. For nonzero v, the cumulative amount of progress x will grow during the time period, so there is likely to be an implicit requirement that the time period's duration Δt not be too long. Thus, situations in which βx^{ε} can be entirely neglected to first approximation might be understood as existing in a small-x and small-Δt limit.

More generally, there are situations in which βx^{ε} can be treated as essentially constant, even if βx^{ε} cannot be viewed as particularly small. Recall that the exponent ε and the rate of innovative progress v have been assumed to be nonnegative. Thus, βx^{ε} never decreases as time elapses. Consequently, to a first approximation the time dependence of βx^{ε} can be treated as negligible if its growth during the relevant time period is much less than $|\alpha + \beta x^{\varepsilon} - \mu|_{\min}$, the minimum magnitude of $\alpha + \beta x^{\varepsilon} - \mu$ during that period. This condition will hold if an upper bound on the growth of βx^{ε} during that period is much less than $|\alpha + \beta x^{\varepsilon} - \mu|_{\min}$. Because at any point in time the rate of change of βx^{ε} equals $\varepsilon\beta x^{\varepsilon-1}(dx/dt)$, which equals $\varepsilon\beta x^{\varepsilon-1}v$, and because $0 < \varepsilon \leq 1$ and therefore $\varepsilon - 1 \leq 0$, an upper bound on the growth of βx^{ε} is given by $\varepsilon\beta x_{\mathrm{i}}^{\varepsilon-1}v_{\max}\Delta t$, where x_{i} is the value of x at the beginning of the relevant time interval, $v_{\max}$ is the maximum speed during that time interval, and Δt is the duration of that time interval. Thus, the time dependence of βx^{ε} would appear to be negligible if $\varepsilon\beta x_{\mathrm{i}}^{\varepsilon-1}v_{\max}\Delta t << |\alpha + \beta x^{\varepsilon} - \mu|_{\min}$, which might be restated more tersely as a requirement that $v_{\max}\Delta t$ be "small enough" under the circumstances. Of course, if one continues to expand the scope of the relevant time interval, this condition will eventually fail to hold, even if $v_{\max}$ does not increase with the expansion. Hence, circumstances for which we can neglect the time dependence of βx^{ε} include small-Δt situations for which the value of x is not required to be particularly small.

When any of the above sets of conditions hold, we can approximate the relevant equation of motion by writing:

$$dv/dt = \alpha' - \gamma v \qquad \text{(Eq. 2)},$$

where $\alpha' = \alpha + \beta x_1^{\varepsilon} - \mu$ or, more simply, $\alpha' = \alpha - \mu$ if we are in a small-x regime where we can ignore the term βx^{ε}.

For students of classical mechanics, Equation 2 might recall the equation of motion for a streamlined object moving slowly through a fluid — so slowly, in fact, that the flow of fluid around the object is essentially laminar (i.e., lacking turbulence).[70] A solution to this equation of motion is the following:

$$v = v_T - (v_T - v_0)e^{-\gamma t} \qquad \text{(Eq. 3)},$$

where v_0 is the speed at time $t = 0$ and v_T is the "terminal velocity" α'/γ.

In words, when Equation 2 applies, the speed of innovation decays exponentially toward a maximum speed $v_T = \alpha'/\gamma$, with the rate of decay equal to the magnitude of the linear-drag coefficient γ. For $t << 1/\gamma$, $e^{-\gamma t} \approx 1 - \gamma t$, and the speed of progress increases approximately linearly:

$$v \approx v_0 + (v_T - v_0)\gamma t \qquad \text{for } t << 1/\gamma \qquad \text{(Eq. 4)}.$$

For $t >> 1/\gamma$ but small enough so that growth of the positive-feedback term βx^{ε} can still be neglected, innovation proceeds essentially at a constant rate, $v_T = \alpha'/\gamma$, and the cumulative sum of technological progress x grows essentially linearly with time t. Indeed, integration of Equation 3 to yield a formula for the amount of cumulative technological progress x yields the following equation:

$$x - x_0 = v_T t - (1/\gamma)(v_T - v_0)(1 - e^{-\gamma t}) \qquad \text{(Eq. 5)},$$

where x_0 is the amount of cumulative technological progress at time $t = 0$.

Hence, if Equation 2 is the appropriate equation of motion for a long enough time period, the result is a linear growth of technology. But as indicated by earlier discussion, the values of t for which linear growth applies can be tightly constrained. If there is any positive feedback — i.e., if $\beta > 0$ — the steady growth of x according to Equation 5 will ultimately undermine the very conditions under which Equations 2, 3, and 5 are understood to apply. Thus, far from being a stable norm, linear or quasilinear growth appears generally to be an unstable condition that contains the seeds of its own destruction. Of course, if technology's advance is viewed as an unambiguous good, that "destruction" is likely to be profoundly "creative," for it coin-

70. *See* GIANCOLI, *supra* note 41, at 259–60 (discussing the motion of a streamlined object through a fluid). *See generally id.* at 249 (defining laminar and turbulent flow).

cides with the onset of more rapid growth reflecting the progress-accelerating effect of growing positive feedback.[71]

For illustrative purposes, Figures 1 and 2 show graphs for the behavior of v and x according to Equations 3 and 5. Please note that these graphs are only illustrative. Under realistic sets of circumstances, the assumptions that underlie Equation 2 might not apply throughout time periods as long as those shown in the graphs — i.e., the value and/or time-dependence of terms such as βx^{ε} and $-\xi v^{1+\eta}$ might not be negligible throughout these periods.

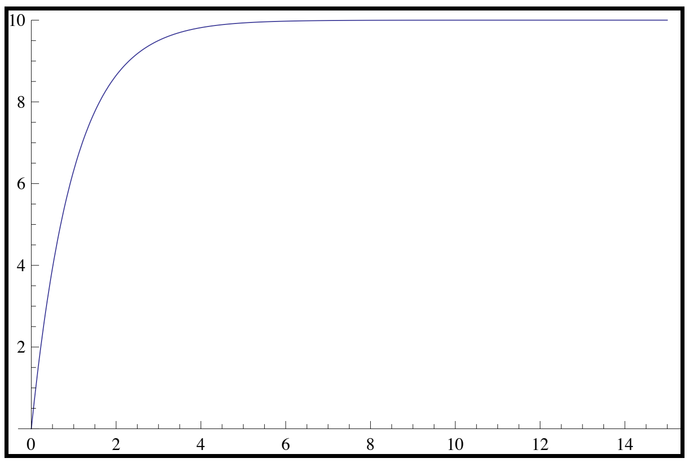

Figure 1: Speed of technological progress v, measured on the vertical axis, as a function of time t, measured on the horizontal axis, for a situation in which Equation 3 applies and, in the units of the graph's axes, $v_0 = 0$, $v_T = 10$, and $\gamma = 1$.

71. *Cf.* JOSEPH A. SCHUMPETER, CAPITALISM, SOCIALISM, AND DEMOCRACY 83 (3d ed. 1950) (discussing "Creative Destruction" as involving "industrial mutation . . . that incessantly revolutionizes the economic structure from within" (footnote and emphasis omitted)).

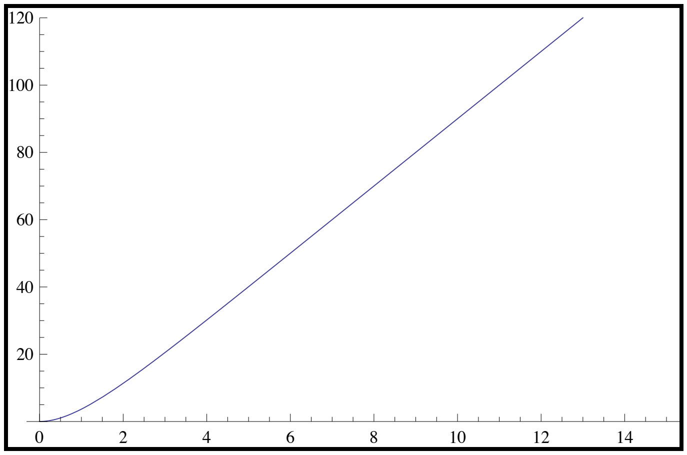

Figure 2: Amount of cumulative technological progress x, measured on the vertical axis, as a function of time t, measured on the horizontal axis, for a situation in which Equation 5 applies and, in the units of the graph's axes, $x_0 = 0$, $v_0 = 0$, $v_T = 10$, and $\gamma = 1$.

2. Exponential Growth for Nonnegligible Linear Positive Feedback

Next, we consider situations where the positive feedback term is linear and nonnegligible:

$$dv/dt = \alpha + \beta x - \mu - \gamma v \qquad \text{(Eq. 6).}$$

As $v = dx/dt$ and $dv/dt = d^2x/dt^2$, Equation 6 can be rewritten as a linear second-order differential equation having the following solution:

$$x = c_1 e^{\kappa t} + c_2 e^{-(\gamma + \kappa)t} - (\alpha - \mu)/\beta \qquad \text{(Eq. 7),}$$

with

$$v = c_1 \kappa e^{\kappa t} - c_2(\gamma + \kappa)e^{-(\gamma + \kappa)t} \qquad \text{(Eq. 8),}$$

$$2\kappa = (\gamma^2 + 4\beta)^{1/2} - \gamma \qquad \text{(Eq. 9),}$$

$$c_1 = \{(\kappa + \gamma)[x_0 + (\alpha - \mu)/\beta] + v_0\}/(2\kappa + \gamma) \qquad \text{(Eq. 10),}$$

and

$$c_2 = \{\kappa[x_0 + (\alpha - \mu)/\beta] - v_0\}/(2\kappa + \gamma) \qquad \text{(Eq. 11),}$$

where x_0 is the value of x at time $t = 0$ and v_0 is the value of v at time $t = 0$. As one would hope, these equations are consistent with Equations 3 and 5 in the limit $\beta \to 0$, in which $\kappa \to 0$ and the positive feedback βx becomes negligible.

Outside the $\beta \rightarrow 0$ limit, we see that, because $\kappa > 0$, Equations 7 and 8 describe functions x and v that are dominated by exponential growth under circumstances where $\alpha + \beta x_0 - \mu > 0$ and the time t satisfies $t >> \{\ln[|c_2|(\kappa + \gamma)/(c_1\kappa)]\}/(2\kappa + \gamma)$.[72] In this large-$t$ limit,

$$x \approx c_1 e^{\kappa t} \qquad \text{(Eq. 12),}$$

and

$$v \approx c_1 \kappa e^{\kappa t} \qquad \text{(Eq. 13).}$$

For purposes of comparison with Figure 2 above, Figure 3 provides a graph of the kind of exponential dependence on time described by Equation 12 for illustrative parameter values.

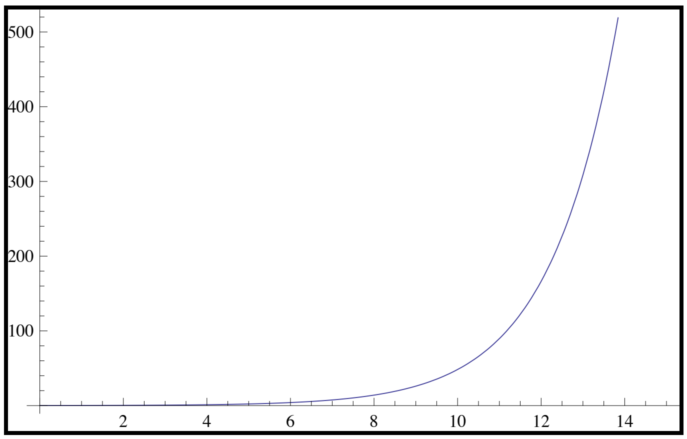

Figure 3: Large-t, leading-order behavior of the amount of cumulative technological progress x, measured on the vertical axis, as a function of time t, measured on the horizontal axis, for a situation in which Equation 12 applies at large values of t and, in the units of the graph's axes, $c_1 = 0.1$, $\gamma = 1$, and $\beta = 1$.

Thus, for nonzero β, linear feedback βx, large t, and negligible supralinear damping $-\xi v^{1+\eta}$, we encounter the exponential growth that Polk Wagner and Brian Arthur predict.[73] On the other hand, just as Part III.A.1's regime of linear growth was shown to contain the seeds of its own destruction, so too is this Part's regime of exponential growth likely to undermine itself with time. Recall that neglect of the higher-order damping term $-\xi v^{1+\eta}$ rested on a requirement that the

72. Given the requirement that v be nonnegative at time zero (as well as all other times), we know that $c_2(\gamma + \kappa) \leq c_1\kappa$ and consequently, because γ and κ are both nonnegative, that $c_2 \leq c_1$. Equations 9 and 10 then follow for $t >> 1/\kappa$, which implies $e^{\kappa t} >> 1$.

73. *See supra* text accompanying notes 49–54.

speed v of technological progress be relatively small.[74] Exponential growth of v according to Equation 13 might rapidly erode satisfaction of this requirement, triggering higher-order damping that, as Part III.B reveals, will result in a more slowly-growing functional form for technological progress. Thus, even when positive feedback is linear and significant, the exponential growth predicted by Polk Wagner and Brian Arthur might not last for long unless public policymakers or private-sector actors can counter the tendency for obstacles to technological progress to scale upward supralinearly with v, for example through supralinearly increasing subsidies or investment.[75]

3. Power-Law Growth for Large Sublinear Positive Feedback

What sort of growth is likely to occur when the positive feedback is sublinear, i.e., subject to diminishing returns? For sublinear feedback, the exponent in the equation

$$dv/dt = \alpha + \beta x^{\varepsilon} - \mu - \gamma v \qquad \text{(Eq. 14)}$$

is less than 1. An equation exhibiting such feedback should not be expected to exhibit long-term exponential growth, for if $x \approx c\mathrm{e}^{\lambda t}$ with $c > 0$ and $\lambda > 0$, then $x^{\varepsilon} \approx c^{\varepsilon}\mathrm{e}^{\varepsilon\lambda t}$, $v \approx c\lambda\mathrm{e}^{\lambda t}$, and $dv/dt \approx c\lambda^2\mathrm{e}^{\lambda t}$. Thus, for sublinear feedback, $-\gamma v$ and dv/dt are the only two terms in Equation 14 that exhibit the highest order of exponential dependence with time, that of $\mathrm{e}^{\lambda t}$. But these terms do not balance each other out. Instead, under the assumption that $x \approx c\mathrm{e}^{\lambda t}$, dv/dt is a positive term on the left-hand side of Equation 14, and $-\gamma v$ is a negative addend on the right-hand side. Thus, under a supposition of exponential growth, the two sides of Equation 14 will have opposite signs at very large times t, making the equality required by Equation 14 an impossibility. Sublinear feedback is therefore not consistent with a supposition of sustained exponential growth according to Equation 14.

Can sublinear feedback sustain power-law growth, in accordance with which, at least in a large-t limit, $x \approx c_3 t^z$ with z being a positive real number? For such power-law growth, $x^{\varepsilon} \approx c_3^{\varepsilon} t^{\varepsilon z}$, $v \approx c_3 z t^{z-1}$, and $dv/dt \approx c_3 z(z-1)t^{z-2}$. The dv/dt term is thus of lower order in t than the $-\gamma v$ term. The βx^{ε} term is the only term that can counterbalance the leading-order behavior of the $-\gamma v$ term and thus yield the required lower-order leading functional dependence for dv/dt. Consequently, in the limit of large positive feedback, we must have the leading-order

74. *See supra* introductory paragraph of Part III.A.

75. Supralinear scaling up of investment in computer chip production might help explain the long reign of Moore's law, in accordance with which chip complexity has grown exponentially for decades. *See* RICHARD N. FOSTER, INNOVATION: THE ATTACKER'S ADVANTAGE 101 (1986) ("The rate of effort (dollars per year) has been increasing even more quickly than chip density.").

formula $\beta x^{\varepsilon} - \gamma v \approx 0$. Calculation then shows that power-law growth can occur in accordance with the following formulas:[76]

$$x \approx c_3 t^{1/(1-\varepsilon)} \qquad \text{(Eq. 15)}$$

and, hence,

$$v \approx [c_3/(1-\varepsilon)]t^{\varepsilon/(1-\varepsilon)} \qquad \text{(Eq. 16),}$$

where

$$c_3 = [\beta(1-\varepsilon)/\gamma]^{1/(1-\varepsilon)} \qquad \text{(Eq. 17).}$$

As we would hope, Equation 15 shows proper limiting behavior. As ε goes to zero and the positive feedback becomes more and more sublinear, the exponents of t in Equations 15 and 16 move closer and closer to 1 and 0, respectively, yielding the limit of linear power-law growth ($z = 1$) and constant terminal velocity that Part III.A.1 has shown to be expected in the absence of positive feedback. On the other hand, as ε goes to one, the exponents of t in Equations 15 and 16 rise dramatically, becoming appropriately ill-defined for $\varepsilon = 1$, the limiting point at which the assumption of power-law growth should break down to make way for the exponential growth described by Equations 12 and 13 above.

For illustrative purposes, consider the results for positive feedback that scales as the square root of x, that is, $\beta x^{1/2}$. Consistent with the argument that positive feedback is likely to show diminishing returns, the slope of $\beta x^{1/2}$ decreases as x becomes larger. For a given infinitesimal increment Δx of technological progress, the added propulsive force for innovation due to positive feedback is approximately $\beta\Delta x/(2x^{1/2})$. Additions to the propulsive force for innovation from positive feedback are thus effectively damped through division by $x^{1/2}$ at large values of x. On the other hand, those additions are still positive, and the speed v of technological progress continues to rise with time. In fact, given that $\varepsilon = 1/2$, Equations 16 and 17 yield $v \approx 2c_3 t$, where $c_3 = [\beta/(2\gamma)]^2$. In other words, v grows essentially linearly with time. It follows that, in a regime where our various approximations apply, the cumulative amount of technological progress grows essentially quadratically — $x \approx c_3 t^2$ — a robust form of power-law growth, even though not exponential. Figure 4 provides a graph of this form of growth for illustrative parameter values.

76. For the various parameter regimes discussed in this article, behaviors predicted through analytical calculation in large-t limits, such as those predicted by Equations 15 through 17, have generally been confirmed to occur through numerical calculation using apparently representative parameter values.

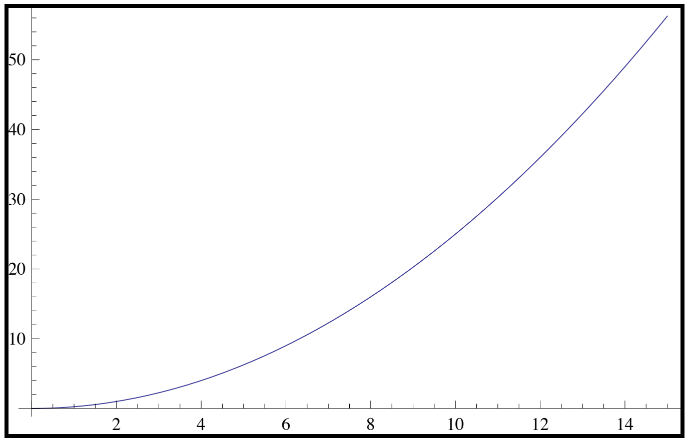

Figure 4: Leading-order, large-t behavior of the amount of cumulative technological progress x, measured on the vertical axis, as a function of time t, measured on the horizontal axis, for a situation in which Equation 15 applies at large values of t and, in the units of the graph's axes, $\varepsilon = 1/2$, $\gamma = 1$, $\beta = 1$, and $c_3 = 1/4$.

Significantly, however, to the extent we have been neglecting the effects of a nonzero $-\xi v^{1+\eta}$ term, such a regime of power-law growth is unstable. This regime will ultimately give way to one of reduced-power-law growth once the speed v of technological progress is large enough that the effects of higher-order damping cannot be ignored. Expected behavior in this comparatively "large-v" regime is the subject of Part III.B.

B. Large-v Dynamics

For large enough values of v, the supralinear damping term $-\xi v^{1+\eta}$ is nonnegligible when compared to the linear damping term $-\gamma v$. For even larger values of v — namely, for $v^{\eta} >> \gamma/\xi$ — the magnitude of the supralinear damping term is much larger than the magnitude of the linear damping term. Because of the possibility that supralinear damping will significantly affect the trajectory of technological progress, the effects of this form of damping are worth investigating.

Further, given the physical analogy between the small-v dynamics described above and the laminar motion of a body through a fluid, it is worth observing that, for $\eta = 1$, the supralinear damping term ξv^2 is quadratic, analogous in form to the common leading-order behavior for the air resistance encountered by a non-streamlined body moving at relatively high speed.[77] Thus, whereas our model provides small-v dynamics analogous to those of a body moving through a fluid in the absence of turbulence, the model provides large-v dynamics that are analogous to those of a body moving through a fluid with turbulence.[78]

1. Terminal Velocity for Negligibly-Variable Positive Feedback

For substantially the same set of circumstances described at the beginning of Part III.A.1 — circumstances under which $\beta = 0$ or, throughout the time period of interest, βx^{ε} is negligibly small or negligibly variant — one can, to first approximation, treat the large-v equation of motion for innovative progress as the following:

$$dv/dt = \alpha' - \gamma v - \xi v^{1+\eta} \qquad \text{(Eq. 18)},$$

where α' is defined as in Part III.A.1 and is thus only time-dependent in the sense that the behavior of μ varies if v goes from a positive value to zero or vice versa.[79]

Like Equation 2 in the limit where the $-\xi v^{1+\eta}$ term is negligible, Equation 18 leads to convergence on a terminal velocity v_T for innovative progress. According to this Part's basic assumptions, the drag coefficient ξ is nonzero and positive, and the drag coefficient γ is nonnegative. It follows that, for $\alpha' \leq 0$, the terminal velocity is zero. Under the circumstances described, dv/dt is always negative until $v = 0$, at which time both α' and dv/dt also go to zero. Hence, even if innovative progress starts out at a pace greater than zero, that progress will slow and eventually grind to a halt without a positive push to counteract velocity-dependent drags.

For $\alpha' > 0$, on the other hand, v_T is a real, positive value that satisfies the equation $0 = \alpha' - \gamma v_T - \xi v_T^{1+\eta}$. When $\gamma = 0$, $v_T = (\alpha'/\xi)^{1/(1+\eta)}$. Thus, for $\eta = 1$ and $\gamma = 0$, $v_T = (\alpha'/\xi)^{1/2}$. Alternatively, for $\eta = 1$ and $\alpha' > 0$, the quadratic formula yields the result:

$$v_T = (\varphi - \gamma)/(2\xi) \qquad \text{for } \eta = 1 \qquad \text{(Eq. 19)},$$

where

$$\varphi = (\gamma^2 + 4\alpha'\xi)^{1/2} \qquad \text{(Eq. 20)}.$$

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77. *See* PAUL A. TIPLER, PHYSICS FOR SCIENTISTS AND ENGINEERS 133 (W.H. Freeman & Co. 4th ed. 1999) (discussing "a drag force of magnitude bv^n, where b is a constant that depends on the shape of the object and the properties of the air, and the exponent n is approximately 1 at low speeds and approximately 2 at higher speeds.").

78. *See* GIANCOLI, *supra* note 41, at 260 ("When turbulence is present, experiments show that the drag force increases as the square of the speed . . . .").

79. *See supra* note 41 and accompanying text.

In fact, for $\eta = 1$ and $\alpha' > 0$, one can derive an exact formula for v evolving in accordance with Equation 18. At time t, the speed v of innovative progress satisfies the equation:

$$(\sigma_- - v)/(\sigma_+ + v) = \Omega e^{-\varphi t} \qquad \text{(Eq. 21)},$$

where

$$\sigma_\pm = (\varphi \pm \gamma)/(2\xi) \qquad \text{(Eq. 22)}$$

and, with v_0 being the speed of innovation at time zero ($t = 0$),

$$\Omega = (\sigma_- - v_0)/(\sigma_+ + v_0) \qquad \text{(Eq. 23)}.$$

Equations 21 through 23 might, by themselves, seem obscure. Their significance is clarified, however, by considering the small-t and large-t limits where $e^{-\varphi t}$ approximately equals 1 or 0, respectively.

In the large-t limit where $t >> 1/\varphi$, the exponential function $e^{-\varphi t}$ approximately equals 0, and v converges on the terminal velocity $v_T = \sigma_-$:

$$v \approx v_T - (\varphi\Omega/\xi)e^{-\varphi t} \qquad \text{for } t >> 1/\varphi \qquad \text{(Eq. 24)}.$$

Thus, in the large-t limit, the speed v of innovative progress converges on the constant value v_T, and the growth of the magnitude x of accumulated progress becomes essentially linear.

Of course, we saw such convergence on a terminal velocity and on linear growth of progress before, when we examined the large-t limit for situations where the time dependence of positive feedback was negligible and the $-\xi v^{1+\eta}$ term could be treated as negligible. This earlier result is connected to that of Equation 24. Indeed, in the limit where $\xi \to 0$, the large-t behavior described by Equation 24 converges exactly to that described by Equation 3, which applies when the supralinear damping term $-\xi v^{1+\eta}$ is negligible.

Similarly, for the small-t limit of $0 \leq t << 1/\varphi$, we have a result that recalls the corresponding outcome for situations where supralinear damping is negligible. In this small-t limit, $e^{-\varphi t} \approx 1 - \varphi t$, and $v \approx v_0$. More specifically:

$$v \approx v_0 + (v_T - v_0)[(1 + v_0/\sigma_+)/(1 + v_T/\sigma_+)]\varphi t \qquad \text{for } t << 1/\varphi \qquad \text{(Eq. 25)}.$$

By comparing Equation 25 with Equation 4, one can see that the small-t behavior for situations where the $-\xi v^{1+\eta}$ term is negligible has much in common with the small-t behavior when the $-\xi v^{1+\eta}$ term with $\eta = 1$ is nonnegligible. In both cases, v initially increases approximately linearly from its value v_0 at $t = 0$, with the rate of linear increase being proportional to $|v_T - v_0|$, the magnitude of the difference between v_T and v_0. Once again, this correspondence is no coincidence:

as ξ goes to zero, φ goes to γ, and σ_+ goes to positive infinity, Equation 25 converges precisely to Equation 4, which applies when supralinear damping can be treated as negligible.

2. Power-Law Growth for Large Linear or Sublinear Positive Feedback

When the time dependence of the positive-feedback term βx^ε cannot be ignored, we again have a situation where the system might be expected to sustain substantially more than linear growth as t becomes very large. Such growth, however, cannot be approximated by an exponential function.[80]

A supposition of power-law growth fares better. As for situations where $\varepsilon < 1$ and, at least effectively, $\xi = 0$, nonnegligible positive feedback can lead x to grow according to a supralinear power law.[81] Suppose that $x \approx c_4 t^z$ where $z > 1$. Then, $x^\varepsilon \approx c_4^{\varepsilon} t^{\varepsilon z}$, $v \approx c_4 z t^{z-1}$, and $dv/dt \approx c_4 z(z-1)t^{z-2}$. Further, $v^{1+\eta} \approx c_4^{1+\eta} z^{1+\eta} t^{(1+\eta)(z-1)}$. The power-law dependence of the $-\xi v^{1+\eta}$ term is therefore necessarily higher than that of any term but the βx^ε positive-feedback term. For the leading behaviors of these terms to cancel — thereby allowing Equation 1 to hold — the following must be true:

$$x \approx c_4 t^{(1+\eta)/(1+\eta-\varepsilon)} \qquad \text{(Eq. 26)}$$

and

$$v \approx c_4[(1+\eta)/(1+\eta-\varepsilon)]t^{\varepsilon/(1+\eta-\varepsilon)} \qquad \text{(Eq. 27)},$$

where

$$c_4 = [(1+\eta-\varepsilon)/(1+\eta)]^{(1+\eta)/(1+\eta-\varepsilon)}(\beta/\xi)^{1/(1+\eta-\varepsilon)} \qquad \text{(Eq. 28)}.$$

---

80. Recall the basic equation for innovation dynamics:

$$dv/dt = \alpha + \beta x^\varepsilon - \mu - \gamma v - \xi v^{1+\eta} \qquad \text{(Eq. 1)},$$

where η is greater than zero and the exponent ε satisfies $0 < \varepsilon \leq 1$. If, to leading order, x grows exponentially at large time scales, we have $x \approx ce^{\lambda t}$ with $c > 0$ and $\lambda > 0$, $x^\varepsilon \approx c^\varepsilon e^{\varepsilon\lambda t}$, $v \approx c\lambda e^{\lambda t}$, $v^{1+\eta} \approx c^{1+\eta}\lambda^{1+\eta}e^{(1+\eta)\lambda t}$, and $dv/dt \approx c\lambda^2 e^{\lambda t}$. Because $1 + \eta > 1$, it follows that, if $x \approx ce^{\lambda t}$, the $-\xi v^{1+\eta}$ term has a higher order of exponential dependence on time than any other term in Equation 1. But this means that the assumption that $x \approx ce^{\lambda t}$ must be wrong at large times. As long as Equation 1 holds and ξ is nonzero, the leading behavior of ξ must be matched by where η is greater than zero and the exponent ε satisfies $0 < \varepsilon \leq 1$. If, to leading order, x grows exponentially at large time scales, we have $x \approx ce^{\lambda t}$ with $c > 0$ and $\lambda > 0$, $x^\varepsilon \approx c^\varepsilon e^{\varepsilon\lambda t}$, $v \approx c\lambda e^{\lambda t}$, $v^{1+\eta} \approx c^{1+\eta}\lambda^{1+\eta}e^{(1+\eta)\lambda t}$, and $dv/dt \approx c\lambda^2 e^{\lambda t}$. Because $1 + \eta > 1$, it follows that, if $x \approx ce^{\lambda t}$, the $-\xi v^{1+\eta}$ term has a higher order of exponential dependence on time than any other term in Equation 1. But this means that the assumption that $x \approx ce^{\lambda t}$ must be wrong at large times. As long as Equation 1 holds and ξ is nonzero, the leading behavior of ξ must be matched by that of some other term or combination of terms.

81. *See supra* Part III.A.3.

Once again, the results exhibit proper limiting behavior. In the limit where $\varepsilon \to 0$, Equation 26 converges to $x \approx c_4 t$, and the constant c_4 converges to $(\beta/\xi)^{1/(1+\eta)}$. Thus, in the $\varepsilon \to 0$ limit, Equations 26 to 28 yield linear-growth behavior with terminal velocity $v_T = (\beta/\xi)^{1/(1+\eta)}$. This is the leading-order behavior expected from Equation 18 in the large-t limit where $\alpha' \approx \beta x^{\varepsilon}$, $\gamma v \approx 0$, and $\varepsilon \to 0$.[82]

For purposes of illustrating the nature of the leading-order behaviors described by Equations 26 to 28, consider the leading large-t behaviors where the supralinear drag term $-\xi v^{1+\eta}$ is quadratic (i.e., $\eta = 1$) and the positive-feedback term βx^{ε} grows either linearly with x (i.e., $\varepsilon = 1$) or as the square root of x (i.e., $\varepsilon = 1/2$). For $\eta = 1$ and $\varepsilon = 1$, the leading-order behavior of the cumulative amount of innovative progress x is quadratic in time, and the leading-order behavior of the speed of innovative progress v is linear: $x \approx [\beta/(4\xi)]t^2$ and $v \approx [\beta/(2\xi)]t$. For $\eta = 1$ and $\varepsilon = 1/2$, the leading-order behaviors of x and v are of positive but lower powers of time: $x \approx [9\beta/(16\xi)]^{2/3}t^{4/3}$ and $v \approx [3\beta^2/(4\xi^2)]^{1/3}t^{1/3}$. Thus, these examples provide a window onto an early stage in the shift toward leading-order behaviors $x \approx v_T t$ and $v \approx v_T$ as $\varepsilon \to 0$. Figures 5 and 6 provide plots of these examples of large-t behavior for x and v for $\eta = 1$, $\varepsilon = 1/2$, and illustrative values for β and ξ.

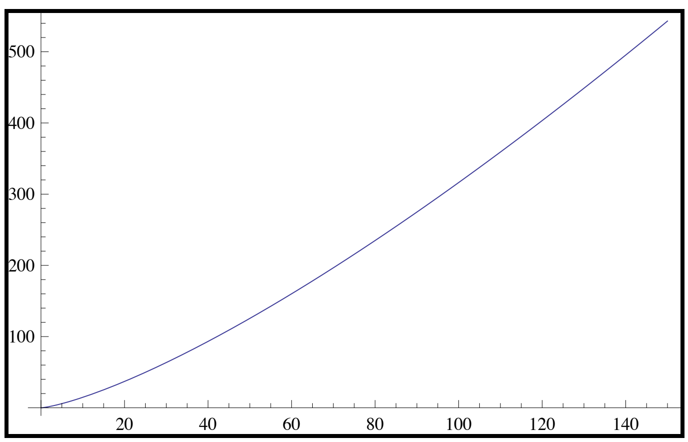

Figure 5: Leading-order, large-t behavior of the amount of cumulative technological progress x, measured on the vertical axis, as a function of time t, measured on the horizontal axis, for a situation in which Equation 26 applies at large values of t and, in the units of the graph's axes, $\varepsilon = 1/2$, $\eta = 1$, $\beta = 1$, and $\xi = 1$.

82. *See supra* Part III.B.1.

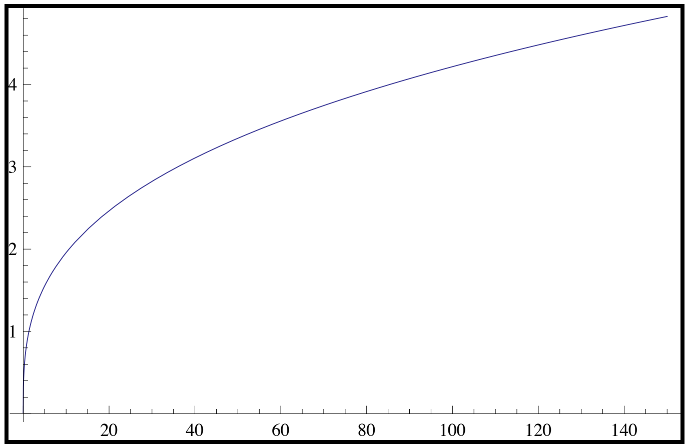

Figure 6: Leading-order, large-t behavior of the speed of technological progress v, measured on the vertical axis, as a function of time t, measured on the horizontal axis, for a situation in which Equation 27 applies at large values of t and, in the units of the graph's axes, $\varepsilon = 1/2$, $\eta = 1$, $\beta = 1$, and $\xi = 1$.

In sum, when the supralinear damping term $-\xi v^{1+\eta}$ is nonnegligible and the power-law dependence of the positive-feedback term βx^{ε} is no more than linear ($0 < \varepsilon \leq 1$), the highest order of growth that we can expect over long time scales is a power-law growth with the cumulative amount of innovative progress developing, to first approximation, according to the formula $x \approx c_4 t^{(1+\eta)/(1+\eta-\varepsilon)}$. On the other hand, similar reasoning to that used above suggests that, if supralinear positive feedback can occur — e.g., if the positive-feedback term βx^{ε} has $\varepsilon > 1$ — leading-order exponential growth can be expected for large t when $\varepsilon = 1 + \eta$.[83] But as indicated above, even the sustainability of

83. For $1 < \varepsilon < 1 + \eta$, large-$t$ power-law growth like that described by Equations 26 to 28 is expected. For $\varepsilon = 1 + \eta$, large-t exponential growth according to $x \approx c_5 e^{\theta t}$ is expected, where $\theta = (\beta/\xi)^{1/\varepsilon}$. For $\varepsilon > 1 + \eta$, on the other hand, Equation 1 appears likely to lead to inverse power-law growth in a large-t limit, with $x \approx c_6(b - t)^{-r}$, where r is a positive real number and b is a large positive real number. The value of b marks the temporal location of a "singularity," in the vicinity of which both the amount of technological progress and the speed of technological progress approach infinity. *See* Ashlee Vance, *Merely Human? That's So Yesterday: The Singularity Movement Sees a Merger of Technology and the Mind*, N.Y. TIMES, June 13, 2010, at BU1 (describing the expectation of "Singularity University founders" of "the Singularity — a time, possibly just a couple decades from now, when a superior intelligence will dominate and life will take on an altered form that we can't predict or comprehend"); Kurzweil, *Law*, *supra* note 54 ("The Singularity is technological change so rapid and so profound that it represents a rupture in the fabric of human history."). The prospect of a theory predicting such a singularity might be another reason to assume a positive-feedback term that is no more than linear in x (i.e., for which $\varepsilon \leq 1$). Alternatively, one might theorize quite plausibly that, at large values of v, an additional higher-order drag term ωv^{ρ}, with $\rho > \varepsilon$, will become significant and cut off the singular behavior

linear $\varepsilon = 1$ feedback is questionable over long time scales. Thus, for the moment at least, there seems cause for confining our attention to situations where the exponent ε of the positive-feedback term βx^{ε} is no more than one.

IV. ANALYSIS OF THEORETICAL RESULTS

A. Different Regimes, Temporal and Technological, for Innovative Progress

Parts IV.B and IV.C discuss specific implications of Part III's results for the development of public policy and patent law in a way that appears most likely to promote innovative progress. More generally, however, Part III's results highlight the fact that, even under a relatively simple model for innovation dynamics, there is no single "natural" trajectory for innovative progress. Thus, it appears wrong to assume that technological progress naturally proceeds exponentially, linearly, or according to some other simple general form with time. Indeed, the exponential growth that some have posited[84] seems a particularly special case — one that lies, as Robert Solow might put it, on a "knife-edge" at which the exponent for the dominant drag term matches the exponent for positive feedback.[85] Far from there being such a generally applicable form, even a simple dynamic model yields multiple distinct "parameter regimes" — defined by ranges of values for (1) variables such as the time t, and (2) externally set parameters such as α, β, and γ — with each regime characterized by its own particular functional forms for the growth and speed of innovative progress.

The theoretical existence of such significantly distinct parameter regimes presents both opportunities and challenges. On the one hand, it provides significant opportunities for empirical work that helps identify the parameter regime in which a category of innovative progress is located or has been located. On the other hand, by highlighting the contingency of even the basic functional behavior of technological progress, the existence of such distinct parameter regimes complicates efforts to determine how technological progress

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that Equation 1's lower-v model predicts. Robert Solow has noted similar divergence concerns with respect to economic growth models that treat technological progress as endogenous. *See* SOLOW, *supra* note 40, at 152 (observing that minor variation of an exponent in such models results in a finite time at which "output blows up to infinity," a result that "does not correspond to common sense").

84. *See supra* text accompanying notes 49-55.

85. SOLOW, *supra* note 40, at 171 (describing a similar situation with respect to models for exponential economic growth that treat technological progress as endogenous); *cf. id.* at 100 ("Such precise assumptions are always hard to justify, and this one usually gets no justification.").

should be measured and how past measurements of technological progress should inform policymaking.

Any particular technological or industrial classification might include a mix of technologies whose developmental trajectories lie in different parameter regimes at a given point in time. The specific boundaries of a technological or industrial classification selected for study might have a significant influence on the nature of this mix, and thus on whether the classification's overall path of development appears dominated by one parameter regime or another. Further, as many of the parameter regimes are unstable in time, the parameter regime of a given technology or set of technologies will likely vary with time. Thus, unless both theory and empirical data are sufficiently developed to enable accurate prediction of a movement from one parameter regime to another, even a well-developed understanding of past behavior might not help much in predicting the trajectory of future development. The technological and economic contingencies that tend to plague patent policy[86] might thus be a common bane of innovation policy generally.

Moreover, movement between parameter regimes during the lifespan of a single technology or set of technologies might only be part of a much larger story of technological progress. A number of commentators have hypothesized that technological progress or its analogs tend to unfold through a time sequence of punctuated equilibria, with spurts of rapid growth interspersed between periods of relatively slow growth or comparative stasis.[87]

Suppose, in accordance with a conventional view, that a single technology or set of technologies progresses in time according to an *S*-shaped curve.[88] This Article's model can account for such a curve as follows. The initial phase of relatively slow advance corresponds to circumstances in which Equation 2 applies and, for some time, growth is at least quasilinear. Positive feedback then becomes a significant

86. *See generally* Golden, *supra* note 17, at 539–50.

87. *See, e.g.*, ARTHUR, *supra* note 19, at 185 (describing how, in a computer simulation of technological development, "periods of quiescence [were] followed by miniature 'Cambrian explosions' of rapid evolution"); OGBURN, *supra* note 49, at 107–08 ("The facts of the growth of material culture seem to indicate a development by jumps. There will be a period of [relative] stability . . . . Then occurs a fundamental invention of great significance which precipitates many changes, modifications and other inventions which follow with relative rapidity for a time. [This is] then followed by another period of relative stability . . . .").

88. *See* CULLIS, *supra* note 30, at 95 ("When a new device is first manufactured, yields are small, but they increase rapidly as efficiency at each process stage improves . . . . After some time, the efficiency approaches an asymptotic value determined by physical constraints."); FOSTER, *supra* note 75, at 98 (observing that the trajectory for improvement of a technology commonly follows an "S-curve" reflecting initial acceleration of progress through "learning" and ultimate deceleration due to "diminishing returns"); *cf.* ANDERSSON & BECKMANN, *supra* note 28, at 55 (observing that research in a new field "generates initially increasing and then decreasing returns, which is characteristic of single-input production activities in general").

driver of innovation, leading to an extended period of rapid technological progress. Finally, a higher-order negative-feedback-informed drag term substantially slows technological growth at large values of x. This might occur because opportunities to develop a particular technology at feasible social cost have essentially been exhausted,[89] or because the hurdles to progress modeled by one or more drag terms have grown increasingly difficult to overcome in a way that overwhelms any progress-promoting positive feedback.[90] A simple *S*-shaped curve with a flat small-time "bottom" and a flat large-time "top" is shown in Figure 7.

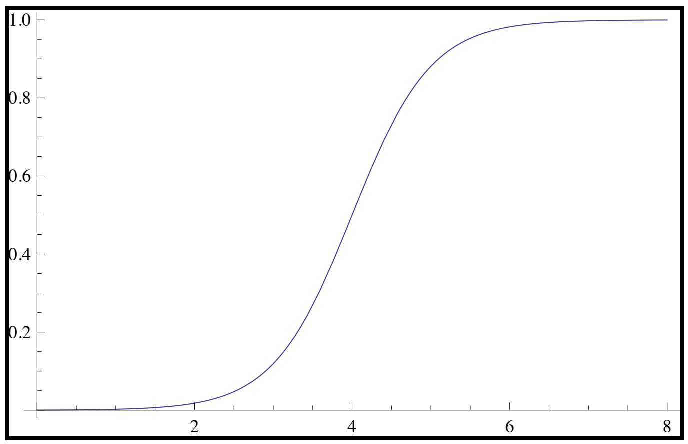

Figure 7: *S*-shaped curve given by the formula $x = [1 + \tanh(t - 4)]/2$, where the time t is expressed in units corresponding to those marking the horizontal axis.

An *S*-shaped curve for technological progress substantially like Figure 7 can be produced by replacing the constant damping coefficient γ in Equation 6, for which supralinear damping is negligible, with the function $\gamma(x)$, where

$$\gamma(x) = \gamma_0/[1 - (x/x_{\max})^q] \qquad \text{(Eq. 29)}$$

89. *See* Carrillo-Hermosilla & Unruh, *supra* note 55, at 709–10 ("Ultimately . . . the increasing returns process is limited by the decreasing returns at the top of the S-curve, which provide the negative feedback that limits exponential growth as markets become saturated."); *cf.* Christoph H. Loch & Bernardo A. Huberman, *A Punctuated-Equilibrium Model of Technology Diffusion*, 45 MGMT. SCI. 160, 168 (1999) (characterizing a "realistic" view of "technology performance changes" as involving "incremental improvements along an *S*-curve," with "[t]he upper limit of the *S*-curve impl[ying] an inherent performance ceiling that a technology cannot surpass").

90. *Cf.* FOSTER, *supra* note 75, at 33 ("[T]echnology even variously defined always has a limit . . . ."); *The Next 20 Years of Microchips*, SCI. AM., Jan. 2010, at 83 (commenting that "Moore's Law" — a "predict[ion] that the complexity of integrated-circuit chips would double every two years" — "could finally be running out of room").

and q is a positive real number. In this modified model, $x_{\max}$ represents a point at which cumulative technological advance must stop — at least so long as the modified model remains valid — as a result, say, of a currently insurmountable obstacle to further technological progress. Under these circumstances, the amount of technological progress x converges asymptotically to $x_{\max}$ from below, with the result being a "flat top" to the technological-progress curve analogous to that shown in Figure 7. Specifically, in the limit where $(x_{\max} - x)/x_{\max} << 1$, the difference between $x_{\max}$ and x decays exponentially with time: $(x_{\max} - x) \approx c_5 e^{-\zeta t}$, where $\zeta = q(\alpha + \beta x_{\max} - \mu)/(\gamma_0 x_{\max})$ and c_5 is a positive constant.[91] Numerical calculation of the trajectory of development for this modified model produces the graph shown in Figure 8 for conditions that, for $x << x_{\max}$, roughly correspond to those used to generate the exponential growth illustrated in Figure 3.

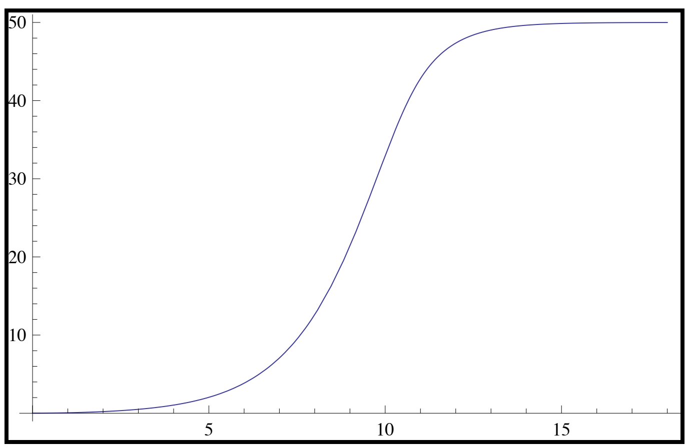

Figure 8: *S*-shaped curve generated by substituting $\gamma(x)$ from Equation 29 for γ in Equation 6, with $\gamma_0 = 1$, $\beta = 1$, $c_1 = 0.1$, $x_{\max} = 50$, $q = 1$, $x_0 = 0$, and $v_0 = 0$.

Of course, obstacles to progress that force a slowdown from a period of rapid growth need not come in the form of a hard "wall." Rather, they might act more like a slowing "bottleneck."[92] Instead of

91. Derivation of the exponential-decay form for the leading behavior of $(x_{\max} - x)$ in the limit where $(x_{\max} - x)/x_{\max} << 1$ is described in the Appendix.

92. The pharmaceutical industry, which has apparently experienced dramatic increases in the costs of developing new drugs, might be experiencing a phenomenon along these lines. *See, e.g.*, Sherry M. Knowles, *Fixing the Legal Framework for Pharmaceutical Research*, 327 SCIENCE 1083, 1083 (2010) ("The costs of drug research and development (R&D) has increased from ~$230 million per drug in the early 1980s to $1.2 billion today, with R&D currently requiring about 10 to 15 years per drug."). Similarly, commentators have long opined that increased costs of progress in semiconductor chip manufacturing might signal that "the technology is beginning to approach its limit." FOSTER, *supra* note 75, at 101; *cf.*

having the drag coefficient $\gamma(x)$ go to infinity at a finite level of progress x_{max}, one could have $\gamma(x)$ simply grow rapidly once x becomes comparable in magnitude to a finite "characteristic value" x_c. For example, one could have

$$\gamma(x) = \gamma_0[1 + (x/x_c)^q] \qquad \text{(Eq. 30)},$$

where q is a positive real number. For such a form of $\gamma(x)$, growth continues beyond the characteristic value x_c, with x ultimately growing (at least while v is still large enough to make the positive-feedback and drag terms dominant) according to the power law $x/x_c \approx (\varsigma t)^{1/q}$ and with v developing according to the power law $v/\varsigma x_c \approx (\varsigma t)^{-1+(1/q)}$, where $\varsigma = q\beta/\gamma_0$. Consequently, for $q > 1$ (i.e., for $-1 + (1/q) < 0$), the speed v of progress decreases after x becomes comparable to x_c.[93] Figure 9 shows a numerically calculated trajectory for growth when Equation 30's $\gamma(x)$ is substituted for γ in Equation 6, under conditions corresponding to those for Figure 8.

CULLIS, *supra* note 30, at 174 ("Capital costs have risen steadily . . . until, currently, the device manufacturers are reaching the stage that they can no longer afford to finance the infrastructure from their individual resources."). Nonetheless, semiconductor chip manufacturing has maintained a trajectory of exponential progress for decades, perhaps partly because positive-feedback effects related to the growth of new markets for computing technology have helped offset the growth in the costs of progress. How long chip manufacturing can continue on this exponential path remains an open question. *See, e.g.*, Antone Gonzales, *Chasing Moore's Law Getting Too Expensive*, INFORMATIONWEEK.COM (June 16, 2009 2:11 PM), http://www.informationweek.com/news/hardware/processors/showArticle.jhtml?articleID=217900161 (reporting a conclusion that "Moore's Law by 2014 will no longer drive volume semiconductor production").

93. The Appendix describes derivation of the power-law leading behavior for x when $x/x_c >> 1$, under circumstances where Equation 6's positive-feedback and drag terms are dominant.

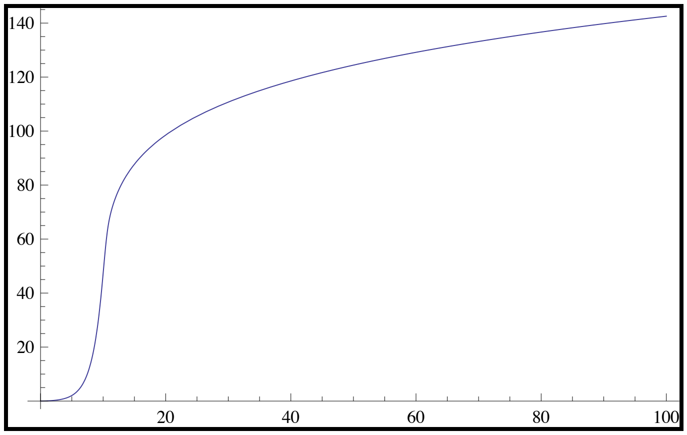

Figure 9: *S*-shaped curve generated by substituting $\gamma(x)$ from Equation 30 for γ in Equation 6, where $\gamma_0 = 1$, $\beta = 1$, $c_1 = 0.1$, $x_0 = 0$, $v_0 = 0$, $x_c = 50$, and $q = 6$.

Whatever the provenance of such an *S*-shaped development path, a time period over which a cycle of *S*-shaped development occurs might be only one episode in a much larger story. The modified version of Equation 6 obtained by substituting x-dependent $\gamma(x)$ for constant γ might cease to be accurate after a further substantial period of time. Faster development might resume, perhaps as a result of a major breakthrough that removes or circumvents the "wall" at x_{max} or the "bottleneck" associated with x_c. The resulting newly-revived trajectory for progress might undergo a new cycle of *S*-shaped growth. In turn, this *S*-shaped cycle might be followed by further spates of *S*-shaped development, with technology alternating between slower and faster phases of advance. Figure 10 shows the sort of overall functional behavior that might result when like cycles of growth are equally spaced in time. This functional behavior might be viewed as embodying a particularly simple pattern of the punctuated equilibria that have been found to characterize the improvement paths of various technologies.[94]

94. *See* MOKYR, *supra* note 58, at 291 ("In the history of technology there are long periods of stasis as well as major discontinuous changes . . . ."); Carrillo-Hermosilla & Unruh, *supra* note 55, at 712 ("[I]f a long enough time horizon is taken, one sees a succession of standards, a dynamic of transition between multiple semi-stable equilibria."); Loch & Huberman, *supra* note 89, at 160 ("In a number of industries it is also observed that long periods of incremental improvement tend to be interrupted by short periods of radical innovation.").

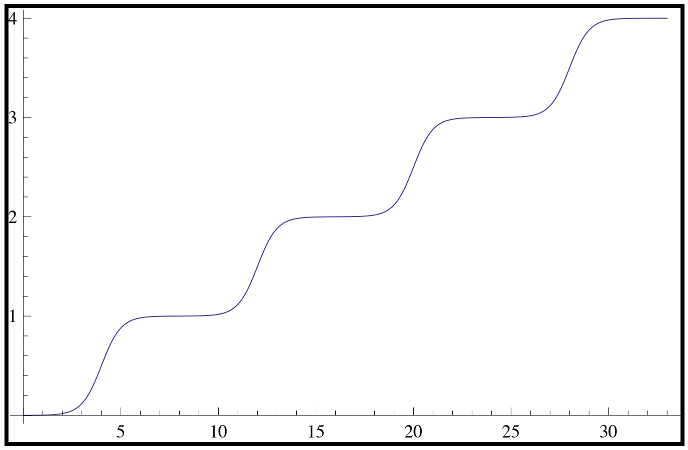

Figure 10: Graph of the sum of four *S*-shaped curves given by the formula $x = \{1 + \tanh[t - 4(2n - 1)]\}/2$, where the time t, is expressed in units corresponding to those marking the horizontal axis, and where n takes on integer values from 1 to 4.

Figure 10 suggests another potential feature of the trajectory of technological progress: that the leading-order behavior over very long time scales might differ greatly from leading-order behaviors observed over much shorter time scales.

For purposes of simplicity, consider a situation like that of Figure 10, in which successive cycles of *S*-shaped growth, each characterized by a "jump" in technological progress of identical height X, are separated by regular time intervals of width T. Over a time interval $t - t_i$ that is much greater than T (i.e., for $(t - t_i)/T >> 1$), the overall amount of technological growth is, to leading order, linear in $t - t_i$, with the progress made from starting position x_i satisfying the approximate formula

$$x - x_i \approx X(t - t_i)/T \qquad \text{(Eq. 31).}$$

For the trajectory of progress shown in Figure 10, $x_i = 0$, $t_i = 0$, $X = 1$, and $T = 8$. Thus, in accordance with Equation 31, $x \approx t/8$ for values of t for which $t/8 >> 1$, presuming that, for such large values of t, the pattern of growth shown in Figure 10 continues to hold.

The dynamics that might be observed at such "super-long" time scales therefore provide a further complication. Super-long time scales can encompass multiple periods of technological progress or stasis. In some circumstances, like those yielding Equation 31 above, complications observed at shorter time scales might substantially average out over super-long time scales, giving way to a much simpler leading-order form for progress's time dependence. In other circumstances, the behavior at super-long time scales might itself exhibit shifts among different behavioral regimes for growth. In any event, the behavior at super-long time scales might be effectively moot for relevant policymakers. The need to satisfy present political needs and the vagaries of super-long-term predictions might be likely to cut off policy horizons at time scales of no more than a few decades.

B. Predictions of Time-Dependent Behavior Compared to Patent Counts

The above discussion of potential behaviors over super-long time scales highlights some of the limitations of the model presented by Equation 1, while also showing how elaborations on the model can overcome some of these limitations. Equation 1's model can still be useful, however, even without such elaborations.

This Part and the one that follows illustrate two ways in which Equation 1 and corollary analysis might be put to practical use. In this Part, the focus is on Equation 1's implications for likely functional forms of technological progress. Part IV.C discusses how dynamic-elasticity or double-ratio tests, derived from Equation 1 for various parameter regimes, might help in addressing questions of innovation policy.

What, beyond contingency and variability, does Equation 1 suggest about the functional form for technological progress? As Part III has shown, the model embodied by Equation 1 indicates that, under a variety of circumstances, technological progress can be expected to grow asymptotically like a positive power of elapsed time, t^z. Such power-law behavior can be looked for in empirical data. If a plausible value for the exponent z is associated with such data, it might permit inferences about the nature of a positive-feedback term like βx^{ε} or the forms of drag terms like γv or $\xi v^{1+\eta}$.

One way to look for power-law behavior is to gather data regarding the value $x(t)$ of a measure of technological progress at different points in time t. One can then examine a log-log plot of that data, graphing the logarithm of progress $s = \ln(x)$ as a function of the logarithm of time $q = \ln(t)$. If technological progress proceeds according to a power law — i.e., if $x(t) \approx Ct^z$ — the log-log plot of technological progress as a function of time should approximate a straight line whose slope equals the power-law exponent z.[95] In contrast, if technological progress proceeds exponentially, a log-log plot yields exponential, rather than linear, behavior.[96]

Different results follow from use of a semi-log plot in which $s = \ln(x)$ is plotted against time t itself, rather than $q = \ln(t)$. An ap-

95. $x(t) \approx Ct^z$ means that $\ln(x) \approx z\ln(t) + \ln(C)$ or, equivalently, $s \approx zq + \ln(C)$.
96. $x(t) \approx Ce^{\kappa t}$ means that $\ln(x) \approx \kappa t + \ln(C)$ or, equivalently, $s \approx \kappa e^q + \ln(C)$.

proximately exponential dependence $x(t) \approx Ce^{\kappa t}$ yields an approximately straight line in a semi-log plot: $s \approx \kappa t + \ln(C)$. If, instead, $x(t)$ exhibits power-law behavior, the natural logarithm of x has only a logarithmic dependence on time: $s \approx z\ln(t) + \ln(C)$. For reference regarding the form of such logarithmic dependence, Figure 11 shows a plot of $s = \ln(t)$.

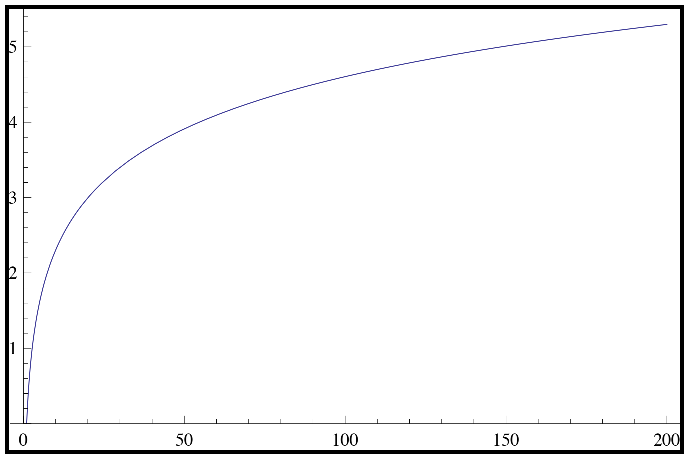

Figure 11: The quantity s as a function of time where $s = \ln(t)$. The form of this graph corresponds to that expected for a semi-log plot of technological progress $x(t)$ where progress has a power-law dependence on time: $x(t) \approx Ct^z$, and thus $\ln(x) \approx z\ln(t) + \ln(C)$.

Log-log and semi-log plots can be used to analyze the time-dependent behavior of a quite crude but easily specified measure of technological progress[97]: the cumulative number of U.S. utility patents that have issued since passage of the U.S. Patent Act of 1790. Figure 12 provides a plot of this cumulative number of patents as a function of time.

Figure 12 makes clear that growth in the cumulative number of patents has accelerated substantially over the course of two centuries. But has this growth been exponential?[98]

97. *But cf.* Olsson, *supra* note 21, at 254 ("The empirical literature on patents and patent citations probably comes closest to actually measuring idea diffusion across time and geography and its impact on the production of new ideas.").

98. *Cf.* Katherine J. Strandburg et al., *Law and the Science of Networks: An Overview and an Application to the "Patent Explosion,"* 21 BERKELEY TECH. L.J. 1293, 1329 (2006) ("[T]he number of patents issued by the USPTO has increased more or less exponentially since the patent system was inaugurated in 1790 . . . .").

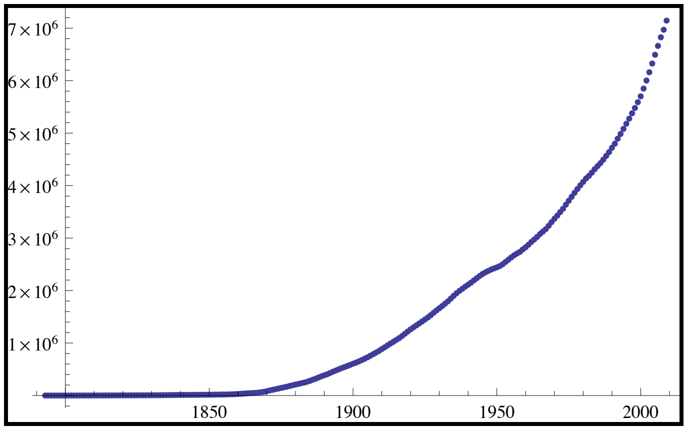

Figure 12: Cumulative number of U.S. utility patents issued from 1790 up to the year indicated on the horizontal axis of the plot.

The log-log and semi-log plots of Figures 13 and 14 suggest that growth has generally not been exponential. Figure 13's log-log plot exhibits substantially linear, rather than exponential, behavior over decades-long time periods, most particularly over a six-decade span following issue of the first utility patents in 1793.

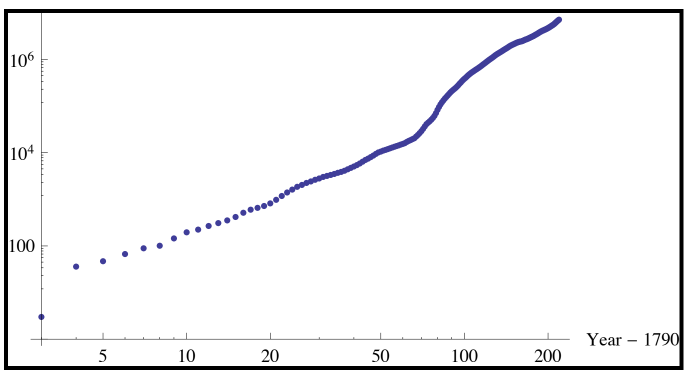

Figure 13: Log-log plot of the cumulative number of U.S. utility patents issued, on the vertical axis, versus the years elapsed since 1790, on the horizontal axis.

Consistent with such impressions from Figure 13, the bowed form of Figure 14's semi-log plot looks very little like the straight-line form expected if the cumulative patent number had grown exponentially since 1790. Instead, Figure 14's bowed form resembles that of $s = \ln(t)$ in Figure 11. Thus, Figures 13 and 14 both suggest that the cumulative number of U.S. utility patents is generally better modeled as having power-law, rather than exponential, time-dependence.

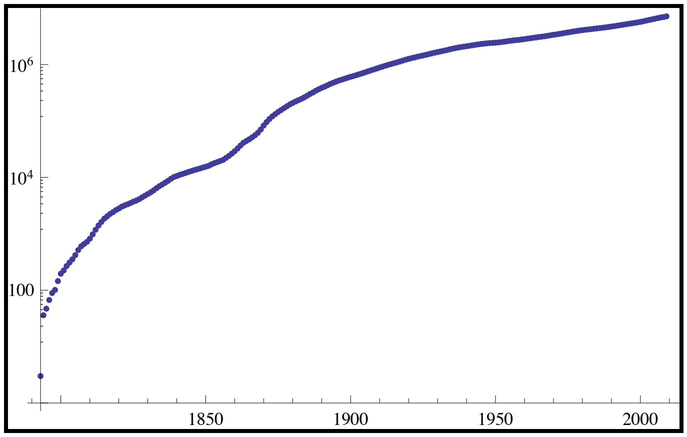

Figure 14: Semi-log plot of the cumulative number of U.S. utility patents issued, on the vertical axis, versus the year, on the horizontal axis.

On the other hand, Figure 13's log-log plot also suggests that, over the course of U.S. history, such power-law behavior has experienced at least two major shifts. An initial phase of patent-count growth ran roughly from 1793 through 1856. As Figure 15 illustrates, a least-squares fit to log-log data from this period yields a line with a slope of about 2.5.[99] This indicates that, during this time period, the cumulative number of U.S. utility patents grew approximately in proportion to $t^{2.5}$ where t is the number of years elapsed since 1790. If we were to boldly assume that technological drag during this period had an approximately linear dependence on the rate of patent accumulation, we might then conclude from Equation 15 that the exponent ε for positive feedback βx^{ε} was approximately 0.6.

99. In Figures 15 through 17, least-squares regression provides linear fits for the accompanying data. Statistical descriptions of the closeness of this fit such as standard errors, R^2 values, and p-values are not reported here, however, in large part because of concern that the use of logarithms of values from the original data sets — times in years and year-by-year cumulative patent counts — might mean that normal bases for attaching specific statistical significance to such descriptions do not apply. *Cf.* Aaron Clauset, Cosma Rohilla Shalizi & M.E.J. Newman, *Power-Law Distributions in Empirical Data,* 51 SIAM REV. 661, 665 (2009) (concluding that the results of "a least-squares linear regression on the logarithm of [a] histogram . . . . cannot be trusted").

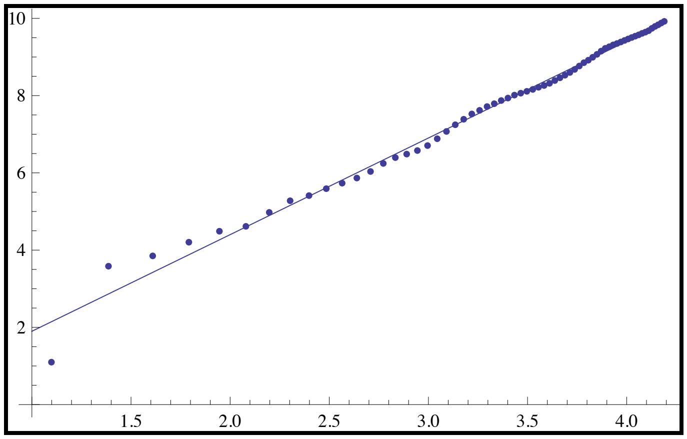

Figure 15: Combined plot of a least-squares regression line and the natural logarithm of the cumulative number of U.S. utility patents issued in the years elapsed since 1790, for the years from 1793 through 1856. The labels on the horizontal and vertical axes indicate logarithmic values for years elapsed since 1790 and for cumulative patent numbers, respectively. The regression line is approximated by the formula $y = -0.6 + 2.5x$.

The second and third growth phases in Figure 13 's log-log plot yield regression lines with higher slopes than that for the first phase. A second phase of distinctively high slope runs from about 1856, the endpoint of the first phase, through 1894. A least-squares fit of log-log data for this period yields a regression line, shown in Figure 16 that has a slope of about 7.3. In other words, during this nearly four-decade period, the cumulative number of U.S. utility patents behaved approximately like a multiple of $t^{7.3}$.

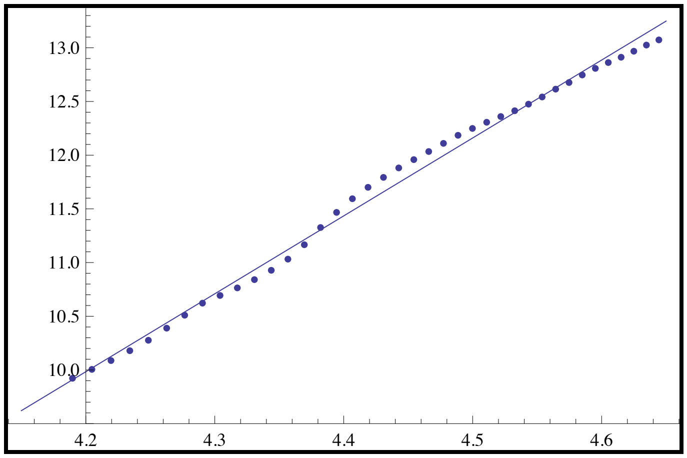

Figure 16: Combined plot of a least-squares regression line and the natural logarithm of the cumulative number of U.S. utility patents issued in the years elapsed since 1790, for the years from 1856 through 1894. The labels on the horizontal and vertical axes indicate logarithmic values for years elapsed since 1790 and cumulative patent numbers, respectively. The regression line is given by the formula $y = -20.5 + 7.3x$.

Finally, Figure 13's third phase of growth runs from about 1894 through 2009. A least-squares fit yields a regression line with a slope of about 3.4, corresponding to a cumulative patent number that behaved, during this time period, approximately like a multiple of $t^{3.4}$. Figure 17 shows the regression line for this third phase, one of approximate power-law growth less robust than that in the second phase but more robust than that in the first.

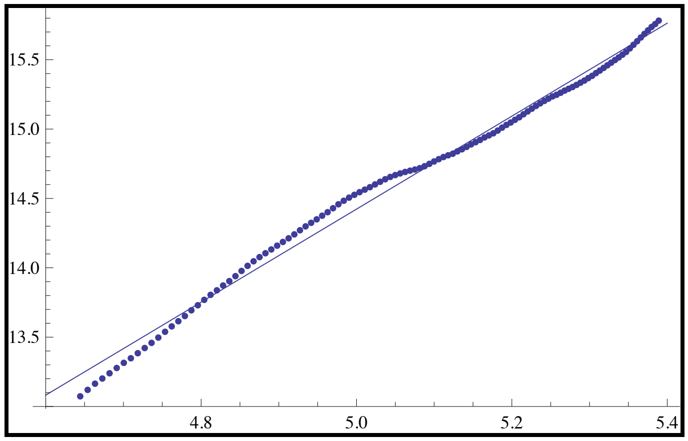

Figure 17: Combined plot of a least-squares regression line and the natural logarithm of the cumulative number of U.S. utility patents issued in the years elapsed since 1790, for the years 1894 through 2009. The labels on the horizontal and vertical axes indicate logarithmic values for years elapsed since 1790 and cumulative patent numbers, respectively. The regression line is approximated by the formula $y = -2.3 + 3.4x$.[100]

More precise statistical analysis might help identify better bounds for distinct phases of U.S. patent-count growth. Likewise, historical study might help explain the location of such bounds, as well as the timing and size of undulations about average behavior. The semi-log plot of Figure 14 might also bear closer scrutiny — for example, with respect to a downward kink that appears to follow shortly on the heels of the U.S. Patent Act of 1836, which moved U.S. patent law from a

100. In accordance with observations that the creation of the U.S. Court of Appeals for the Federal Circuit in the early 1980s was followed by a sharp rise in patenting, *see supra* note 9; *see also* Strandburg et al., *supra* note 98, at 1329 ("[T]he rate of increase [in issued patents] . . . sharpened noticeably in the early 1980s."), the plot in Figure 17 appears to show an upward inflection in the cumulative patent count at about the year 1983 — a time corresponding to a point just beyond the 5.26 mark on Figure 15's horizontal axis. But this upward inflection does not yet appear to have led to a deviation from approximate power-law behavior beyond what might have been expected based on long-term trends. Likewise, the "Information Age" of the last few decades, Bilski v. Kappos, 130 S. Ct. 3218, 3228 (2010) (characterizing the present "Information Age" as "put[ting] the possibility of innovation in the hands of more people and rais[ing] new difficulties for the patent law"), does not yet appear to have effected a change in the time-dependent behavior of cumulative patent numbers comparable to that which occurred during the latter half of the nineteenth century. *Cf.* Carol A. Corrado & Charles R. Hulten, *How Do You Measure a "Technological Revolution"?*, 100 AM. ECON. REV. 99, 102 (2010) (reciting economist "Robert Solow's famous quip that 'you see the computer age everywhere except in the productivity data'").

registration system to an examination system.[101] But such projects go beyond this Article's scope. The example of patent-count data is illustrative, showing how future empirical work might look for power-law or exponential behavior, and how it might relate such behavior to Equation 1's model.

From the standpoint of Equation 1's model, it is a bonus that, over decades-long time scales, the historical development of U.S. utility patent numbers appears remarkably consistent with power-law behavior, as the model might lead one to expect. The simplifications and assumptions that underlie Equation 1 might have rendered such success doubtful. For example, if one had believed Equation 1's assumption of constant values for the exponents ε and η to be implausible, the relative constancy of the power-law dependence seen in Figures 15 through 17 might well have been a surprise. Will other measures of technological progress — such as technical measures of the average speed of communications or the efficiency of electrical energy generation — reveal similar surprises? Future research might look to answer this question.

C. Dynamic-Elasticity Tests for Questions of Innovation Policy

This Part describes a second way in which Equation 1's model for innovation dynamics contributes to the understanding of when particular policies are likely to promote innovative progress. The model does this by indicating the potential significance of what I call dynamic-elasticity or double-ratio tests for whether a particular incremental policy change will promote progress.

The terminology "double-ratio tests" is meant to distinguish such tests from ratio tests for patent-related policy that Louis Kaplow described in 1984.[102] According to such a "single-ratio test," the relative desirability of a particular policy — such as permitting patentee licensing practices that might raise antitrust concerns — can be assessed by considering the ratio between the additional patentee rewards associated with the policy and the additional social costs associated with the policy.[103] An even more fundamental ratio test —

101. F. SCOTT KIEFF ET AL., PRINCIPLES OF PATENT LAW 20 (4th ed. 2008) ("The registration system lasted for 43 years, until July 4, 1836, when Congress enacted what is generally acknowledged to be the foundation of the modern patent system in the United States.").

102. *See* Louis Kaplow, *The Patent-Antitrust Intersection: A Reappraisal*, 97 HARV. L. REV. 1813, 1816 (1984) (describing a "ratio test" to address "the patent-antitrust puzzle"); *see also* Daniel A. Crane, *Intellectual Liability*, 88 TEX. L. REV. 253, 272 (2009) (describing Kaplow's "patentee-reward to monopoly-loss ratio" test and contending that it "is useful for appraising the appropriate sticks in the bundle of intellectual rights more generally").

103. *See* Kaplow, *supra* note 102, at 1834 (suggesting that social welfare can be improved by "shuffling the extant pattern of [antitrust] restrictions" by, for example, disfavoring "a currently permitted practice with a low ratio" in comparison to "a currently prohibited practice with a high ratio").

also described by Kaplow, albeit without specific formulation as a ratio test — is that, for any particular policy to be socially desirable, the ratio between the "marginal social benefit" generated by the policy and the policy's "marginal social cost" should be greater than or equal to one (where the marginal social cost is here assumed to be positive).[104] Thus, a ratio test as it has commonly been understood involves a ratio of the general form:

$$\Delta B/\Delta C \qquad \text{(Eq. 32)},$$

where ΔB is an incremental benefit associated with the policy in question and ΔC is an incremental cost associated with the policy. Such a single-ratio test tends to indicate either that the policy change in question is desirable if $\Delta B/\Delta C > 1$, or at least that the policy change in question is relatively more desirable as $\Delta B/\Delta C$ becomes larger.[105]

The double-ratio tests presented here involve two ratios having the forms $\Delta B/B$ and $\Delta C/C$, respectively, where ΔB is the incremental change (positive or negative) that the policy produces in a positive parameter B that is associated with faster technological progress, and ΔC is the incremental change that the policy produces in a positive parameter C that is associated with slower technological progress. A double-ratio test using these two ratios indicates that, at least to a first approximation, an incremental policy change is desirable if

$$\Delta B/B > \Delta C/C \qquad \text{(Eq. 33)}.$$[106]

When the ratio $\Delta C/C$ is positive, this double-ratio test can be written in the alternative form $(\Delta B/B)/(\Delta C/C) > 1$. The double ratio $(\Delta B/B)/(\Delta C/C)$ might be viewed as a form of "dynamic elasticity" — a ratio of the percentage "dynamic benefit" increase due to a new policy to the percentage "dynamic cost" increase as a result of that policy. Thus, the double-ratio test can be described as a form of dynamic-elasticity test.[107]

Dynamic-elasticity or double-ratio tests arise from Part II's model because a number of the results from Part III indicate that the leading-order, large-time behavior of innovative progress depends heavily on a ratio of push and drag parameters. For example, Equation 3 in Part III.A.1 indicates that, for sufficiently large times and for conditions

104. *Cf. id.* at 1825 ("The optimal patent life is that length of time at which the marginal social cost of lengthening or shortening the patent life equals the marginal social benefit.").

105. *See supra* notes 102–03.

106. For purposes of terminological and notational simplicity, this Article commonly refers to inequalities such as that of Equation 33 as equations.

107. Economists commonly use the word "elasticity" to refer to a ratio of percentage changes in two quantities. *See, e.g.*, WILLIAM J. BAUMOL & ALAN S. BLINDER, ECONOMICS: PRINCIPLES AND POLICY 465 (5th ed. 1991) (providing an equation for the "elasticity of demand" (emphasis omitted)); HAL R. VARIAN, INTERMEDIATE MICROECONOMICS 265–66 (3d ed. 1993) (same).

under which the supralinear damping term $-\xi v^{1+\eta}$ and the time dependence of the positive-feedback term βx^{ε} are effectively negligible, we have $v \approx v_T$, with

$$v_T = \alpha'/\gamma \qquad \text{(Eq. 34)},$$

where α' and γ are positive real numbers. For circumstances where $v \approx v_T$ applies at large times, a policy change will tend to promote progress over such a time scale if the policy change tends to increase the quantity v_T — i.e., if the change Δv_T in v_T that results from the policy change is greater than zero. Where the policy change produces changes $\Delta\alpha'$ and $\Delta\gamma$ in the quantities α' and γ, respectively, Δv_T is given by the equation $\Delta v_T = (\alpha' + \Delta\alpha')/(\gamma + \Delta\gamma) - \alpha'/\gamma$. Alternatively stated, $\Delta v_T/v_T = (\Delta\alpha'/\alpha' - \Delta\gamma/\gamma)/(1 + \Delta\gamma/\gamma)$. As long as we make the relatively trivial presumption that the policy change does not reduce the drag coefficient γ to zero — i.e., that $\gamma + \Delta\gamma > 0$, and thus that $1 + \Delta\gamma/\gamma > 0$ — it follows that the sign of Δv_T is the same as the sign of the quantity $(\Delta\alpha'/\alpha' - \Delta\gamma/\gamma)$. Consequently, we have the following double-ratio test for whether a policy change is likely to promote the speed of innovative progress:

$$\Delta\alpha'/\alpha' > \Delta\gamma/\gamma \qquad \text{(Eq. 35)}.$$

A key distinction between such a dynamic-elasticity or double-ratio test and a single-ratio test involving a ratio like that of Equation 32 is that, under the double-ratio test, the key determinants of the policy's desirability are not so much the raw magnitudes of the expected changes in progress-promoting or progress-retarding terms, but instead the expected percentage changes in those terms. Thus, even when a policy produces only a seemingly small positive impetus to innovation while also introducing seemingly substantial contributions to innovative drag, that policy might nonetheless help speed progress. This can result when the pre-policy impetus to innovation, as reflected in the progress-forcing term α', is very small and pre-policy impediments to innovative progress, as reflected in the drag coefficient γ, are already comparatively large.

This distinguishing aspect of double-ratio tests has important implications. In particular, the double-ratio test suggests that, in industries such as the pharmaceutical industry, where regulation and safety concerns help generate large innovative drag that exists independently of the details of patent law, comparatively stronger or broader patent protection might speed progress even if such increased protection imposes substantial costs on follow-on innovators. This follows because whatever increases these costs make to the drag coefficient γ might be negligible compared to γ's preexisting magnitude. Consequently, the percentage change in γ from the increased costs might be small, and even if the policy change produces only a relatively small percentage uptick in the progress-forcing term α', that small increase might justify the follow-on costs.

On the other hand, if a technology (such as software) would be characterized, in the absence of patents, by both strong incentives for innovation and relatively low costs for follow-on innovation, the analysis might be reversed. For such a technology, the value of the progress-forcing term α' might already be very large before any policy change is implemented. Such a situation might arise under circumstances where user innovation is a prevalent or even dominant mode of technological progress.[108] Under such circumstances, unless $\Delta\alpha'$, the further impetus to progress provided by the policy change, is extremely large, even a modest increase in γ might mean that the policy change does more to impede progress than to promote it.

More generally, comparison of the analyses in the two preceding paragraphs might do much to explain the differing positions of various industries and industry actors in recent patent reform debates. Companies in technology sectors such as software or financial services, in which there are, apparently, relatively low costs for follow-on innovation and strong incentives for innovation even without patent rights,[109] have been looking to curtail what they view as overextensions of patents' effective strength or scope.[110] Companies in technology sectors such as pharmaceuticals, in which regulatory or other hurdles likely make a drag coefficient such as γ extraordinarily high, have tended to defend the strength or scope of patent rights.[111] Entrepreneurial biotechnology firms facing large start-up costs and having few non-patent mechanisms for appropriating value from innovation[112] might similarly experience low-α' and high-γ conditions that make stronger or broader patents desirable in their technology sector.

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108. User innovation is frequently associated with significant progress-promoting pushes that operate without the drags associated with the need for commercial profit from an invention. *See* Katherine J. Strandburg, *Evolving Innovation Paradigms and the Global Intellectual Property Regime*, 41 CONN. L. REV. 861, 875 (2009) ("In sharp contrast to the standard seller-based view underlying most discussions of the societal justifications for the patent system, user innovators expect to benefit primarily from developing and using an innovation rather than selling it."); *cf.* ERIC VON HIPPEL, DEMOCRATIZING INNOVATION 14 (2005) (associating the proliferation of user innovation in design with "the cost of high-quality resources for design and prototyping becom[ing] very low"); William W. Fisher III, *The Implications for Law of User Innovation*, 94 MINN. L. REV. 1417, 1423 (2010) ("Purchasers of sports equipment frequently alter the equipment to fit their bodies or needs.").

109. These conditions correspond to a relatively high value for α' and a relatively low value for γ.

110. *See* Golden, *supra* note 17, at 507–08.

111. *Cf. id.*

112. *See* Stuart J.H. Graham et al., *High Technology Entrepreneurs and the Patent System: Results of the 2008 Berkeley Patent Survey*, 24 BERKELEY TECH. L.J. 1255, 1308 (2009) (reporting on the apparent importance of patents for private investment in biotechnology firms).

Even if a double-ratio test indicates the desirability of a policy change for one technology sector, however, there would remain difficult questions of how a policymaker should weigh the potentially disparate impacts of the policy change on different technology sectors' progress — questions that no mathematical model is likely to answer, except by incorporating value judgments that the model itself does not justify. Nonetheless, it seems no small advance to arm policymakers with a dynamic model that provides new means of (1) understanding when and how a policy change might affect different sectors disparately, and (2) potentially gaining insight into how to mitigate negative impacts while pursuing a positive policy end — for example, by comparing different policies' likely impacts on push and drag parameters.

Along these lines, it might be instructive to consider potential implications of double-ratio tests for the categories of subject matter that might be considered patentable. A double-ratio test might support the imposition of limits on such categories. For example, courts have traditionally held that claimed inventions that amount to no more than "abstract ideas" or "natural phenomena" lack patentable subject matter.[113] A double-ratio test can help justify both of these exclusions, as well as their relatively narrow nature.

Consider the exclusion from patentable subject matter of laws of nature and natural phenomena,[114] which might be understood as seeking to reserve a relatively patent-free zone for basic scientific research. Even without patent rights as an incentive, development of basic scientific knowledge about natural phenomena is stimulated by a robust combination of public funding, prizes, reputational rewards, opportunities for career advance, and funding from private interests motivated by first-mover advantages, altruism, or other causes.[115] In this context, the quantity α' in the double-ratio test is likely to be substantial for a wide range of scientific endeavors. Meanwhile, in the absence of patent rights, there might be relatively little innovative

113. *See* Diamond v. Diehr, 450 U.S. 175, 185 (1981) ("Excluded from . . . patent protection are laws of nature, natural phenomena, and abstract ideas.").

114. *Id.*

115. *See, e.g.*, Dan L. Burk & Mark A. Lemley, *Policy Levers in Patent Law*, 89 VA. L. REV. 1575, 1586 (2003) ("Inventors may be motivated by the prospect of prestige among peers, by prizes (such as the Nobel)[,] . . . by academic rewards of promotion and tenure[,] . . . by the desire to do good, . . . or by the love of science."); Rebecca S. Eisenberg, *Proprietary Rights and the Norms of Science in Biotechnology Research*, 97 YALE L.J. 177, 182–84 (1987) (describing a Mertonian "conception of the norms and incentives that guide the behavior of research scientists," including "recognition and esteem"); John M. Golden, *Biotechnology, Technology Policy, and Patentability: Natural Products and Invention in the American System*, 50 EMORY L.J. 101, 190 (2001) (noting evidence "that a key aspect of biotechnology is the fact that, to an exceptionally large degree, it results from a massive public investment both in the life sciences and in the training of life scientists"); Arti Kaur Rai, *Regulating Scientific Research: Intellectual Property Rights and the Norms of Science*, 94 NW. U. L. REV. 77, 92 (1999) ("The norm of invention leads scientists to compete vigorously to be the first to present an invention or discovery to the scientific community.").

drag: with commercial applications frequently far in the distance, there might be few regulatory hurdles to clear or little need to spend time and energy packaging results in a manner suitable for mass use by consumers. The quantity γ in the double-ratio test might therefore be relatively small. Given these expectations regarding background pushes and drags, the double-ratio test suggests that an increased presence of patent rights in the subject matter of basic science will slow down, rather than speed up, the rate of progress — unless patents will (1) add greatly to incentives for developing basic scientific knowledge and (2) add very little to the costs of subsequent scientific research.

Similar arguments in relation to the drag side of an applicable double-ratio test appear likely to extend to the excluded category of abstract ideas. On the other hand, arguments with respect to the push side of the test might be more balanced for abstract ideas in general. The category of abstract ideas is so broad and polymorphous that one might wonder whether there are segments of the category for which society currently provides little effective incentive. Indeed, despite the existence of factors such as principles of academic freedom that help to ensure that scientific research is wide-ranging and decentralized, this concern might extend to some forms of basic scientific knowledge as well. In the 1990s, prohibition of federal public funding for human embryonic stem cell research arguably created such a situation in the United States, and the prospect of patent rights in the fruits of research helped generate crucial initial funding where conventional public-funding mechanisms were inoperative.[116] Given the possibility that incentives for the development of abstract ideas and even basic science might sometimes be very weak, the double-ratio test does not unambiguously support a broad understanding of the traditional judge-made exclusions.

A predictable result of such ambiguity is the sort of hedge that courts have tended to adopt: judge-made rules on patentable subject matter generally forbid patent rights for natural laws, natural phenomena, and abstract ideas, but at the same time impose only relatively minimal requirements for circumventing these exclusions. These minimal requirements have included (1) tying the exploitation of an abstract idea to a human-made device or a transformation of matter;[117]

116. *See* John M. Golden, *WARF's Stem Cell Patents and Tensions Between Public and Private Sector Approaches to Research*, 38 J.L. MED. & ETHICS 314, 314–15 (2010).

117. *See In re* Bilski, 545 F.3d 943, 954 (Fed. Cir. 2008) (en banc) (holding that the United States Supreme Court had "enunciated a definitive test" for the subject-matter eligibility of a process, centered on the question of whether the claimed process "(1) . . . is tied to a particular machine or apparatus, or (2) . . . transforms a particular article to a different state or thing"), *aff'd sub nom.* Bilski v. Kappos, 130 S. Ct. 3218 (2010). *But see* Bilski v. Kappos, 130 S. Ct. at 3227 ("The machine-or-transformation test is not the sole test for deciding whether an invention is a patent-eligible 'process.'").

and (2) creating an isolated and purified version of a naturally occurring substance.[118] Better understanding of innovation dynamics and likely tradeoffs in promoting progress might facilitate the development of better-tailored and better-justified boundaries for patentable subject matter. In the meantime, the potential ambiguity of the double-ratio test across the full breadth of abstract ideas or basic science might be viewed as supporting imposition of something less than insurmountable hurdles to patenting related subject matter. Although perhaps theoretically inelegant, such reduced hurdles might leave open a path — a sort of "safety valve"[119] — for patents to promote potentially significant basic research or abstract thinking in situations where alternative modes of motivation fail.

Whatever the theoretical utility of a dynamic-elasticity or double-ratio test, one might question whether the particular double-ratio test of Equation 35 applies, under present or expected conditions, to any given technology or industry sector. One response to such skepticism is to note that, even if Equation 35's double-ratio test does not apply, an alternative double-ratio test might. Further examination of the behaviors predicted in Part III.A yields additional dynamic-elasticity or double-ratio tests and establishes the circumstances under which they apply. What follows is a list of additional tests for situations in which (1) parameters such as α', β, γ, and ξ are assumed to be positive, and (2) exponent-based parameters such as ε and η in Equation 1 are assumed to be unaffected by the policy change in question:

- $\Delta\beta/\beta > \Delta\gamma/\gamma$, where the following conditions apply: (i) in accordance with Equation 6, growth is exponential due to the positive-feedback term βx being linear and the supralinear damping term $-\xi v^{1+\eta}$ being negligible; (ii) $4\beta/\gamma^2 << 1$; and (iii) the large-t limit where $t >> \{\ln[|c_2|(\kappa + \gamma)/c_1\kappa]\}/(2\kappa + \gamma)$ and $t >> \ln[c_1/(c_1 + \Delta c_1)]$ applies, where Δc_1 is the incremental change in c_1 due to the policy change;
- $\Delta\beta/\beta > \Delta\gamma/\gamma$, where (i) the positive-feedback term βx^ε is sublinear, (ii) the supralinear damping term $-\xi v^{1+\eta}$ is negligible, and

118. *See* MERGES & DUFFY, *supra* note 11, at 113 (discussing how, under current law, a naturally occurring substance existing "as it is produced and used in the human body is viewed as different from the substance in its isolated and purified state"); *cf.* Amgen, Inc. v. Chugai Pharm. Co., 927 F.2d 1200, 1206 (Fed. Cir. 1991) (emphasizing that "[t]he subject matter of [a patent claim] was the novel *purified and isolated* [genetic] sequence which codes for" the protein erythropoietin). *But see* Ass'n for Molecular Pathology v. U.S. Patent & Trademark Office, 702 F. Supp. 2d 181, 227 (S.D.N.Y. 2010) (concluding that "purification of a product of nature, without more, cannot transform it into patentable subject matter").

119. *Cf.* Henry E. Smith, *Institutions and Indirectness in Intellectual Property*, 157 U. PA. L. REV. 2083, 2127 (2009) (arguing, in relation to debates over patent–infringement injunctions, for viewing "equitable analysis as the correct safety valve for enriching the interface between otherwise quite modular rights").

(iii) $x \approx c_3 t^{1/(1-\varepsilon)}$ in accordance with Equation 15 and the appropriate large-t limit for Equation 14;

- $\Delta\alpha'/\alpha' > \Delta\xi/\xi$, where (i) $\eta = 1$; (ii) Equation 18 applies because the supralinear damping term $-\xi v^2$ is nonnegligible but the positive-feedback term βx^ε, or at least its time dependence, is negligible; (iii) the time $t >> 1/\varphi$, and innovative progress is therefore essentially proceeding at the constant terminal velocity $v_T = \sigma_-$; and (iv) $\gamma/[2(\alpha'\xi)^{1/2}] << 1$;
- $\Delta\alpha'/\alpha' > \Delta\gamma/\gamma$, where, as immediately above, (i) $\eta = 1$; (ii) Equation 18 applies because the supralinear damping term $-\xi v^2$ is nonnegligible but the positive-feedback term βx^ε, or at least its time dependence, is negligible; and (iii) the time $t >> 1/\varphi$, and innovative progress is therefore essentially proceeding at the constant terminal velocity $v_T = \sigma_-$; but where (iv) $\gamma^2/(4\alpha'\xi) >> 1$;
- $\Delta\beta/\beta > \Delta\xi/\xi$, where (i) both the supralinear damping term $-\xi v^{1+\eta}$ and the time dependence of the positive-feedback term βx^ε are nonnegligible, with $0 < \varepsilon \leq 1$; and (ii) the large-t limit of Equation 1 in which, in accordance with Equation 26, $x \approx c_4 t^{(1+\eta)/(1+\eta-\varepsilon)}$ applies.[120]

As might be expected, the "push ratio" on the left-hand side of each inequality represents the percentage change expected for a parameter, α' or β, associated with one or more of the progress-forcing terms of Equation 1. In turn, the "drag ratio" on the right-hand side represents the percentage change expected for a parameter, γ or ξ, associated with one of Equation 1's velocity-dependent damping terms.

Significantly, the parameters that are the focal points of each double-ratio test reflect the terms in Equation 1 that dominate innovation dynamics under the particular circumstances in question. Thus, in a large-time limit where positive feedback dominates the ongoing "push" for innovation, one can have a double-ratio test such as $\Delta\beta/\beta > \Delta\gamma/\gamma$. When applicable, such a test suggests that the extent to which a policy change adds to innovative push without directly enhancing positive feedback — i.e., by increasing the push parameter α but not the positive-feedback coefficient β — will commonly be less important, over relatively long time scales, than the extent to which the policy change facilitates positive feedback. For example, to the extent that providing grant money for technological development is associated with increasing α but not β, the applicability of a $\Delta\beta/\beta > \Delta\gamma/\gamma$ double-ratio test might suggest that such a policy should be disfavored relative to one that looks likely to have a greater posi-

120. The Appendix derives these further dynamic-elasticity or double-ratio tests and explains the circumstances under which they apply.

tive impact on β — perhaps by encouraging relevant firms' reinvestment of profits in research and development.

In any event, despite constraints on when the various double-ratio tests apply, the existence of a multiplicity of such tests that apply under distinct sets of conditions shows that the utility of a double-ratio test is not an isolated phenomenon. Further, as Kaplow showed earlier for single-ratio tests, such tests' contributions to analysis and intuition can be substantial even when the precise values of their inputs are uncertain.[121] The preceding discussion of how double-ratio tests can help explain current industry and technology divides indicates how double-ratio tests can be qualitatively useful.

V. CONCLUSION

A relatively simple but intuitively plausible model of innovation dynamics indicates that there is likely to be no "natural" form for the trajectory of technological progress. Instead, the model generates a diverse array of potential trajectories, among which trajectories featuring linear or exponential growth of cumulative innovation are only special or even exceptional. Different parameter regimes, involving shorter or longer time scales and different values for progress-promoting and progress-impeding parameters, yield different forms for progress's leading-order behavior. In certain regimes, technological advance proceeds approximately linearly with time. In others, growth proceeds according to a supralinear power law. In a comparatively narrow class of situations, growth proceeds exponentially. In still other situations, a technological "wall" or "bottleneck" causes growth to slow dramatically after a certain amount of progress is made, with the result being an *S*-shaped trajectory for progress.

These complexities follow from the tension between the dynamic model's progress-promoting and progress-impeding terms. The progress-promoting "push" terms can reflect a significant positive-feedback effect that causes accelerants for technological progress to grow as technology advances. The progress-impeding "drag" terms feature negative-feedback effects due to, for example, the fact that the faster technology progresses, the greater the number of obstacles that innovators must overcome in any given time period. Because push and drag terms commonly scale differently with time, even Part II's relatively simple model for innovation dynamics yields a rich array of contingent behaviors.

121. *See* Kaplow, *supra* note 102, at 1886–87 ("Although one can neither state with certainty that permitting end-product royalty schemes improves the ratio, nor confidently identify the range of circumstances under which it might not, the argument for permitting them does seem stronger than that for disallowing them.").

For policymakers, a significant implication of this Article's analysis is that, even if they can identify the current behavioral form for technological progress, they should not generally assume that such a form will persist. As Part III's analysis of different parameter regimes shows, a given functional form for technological progress often sows the seeds of its own destruction, sometimes by feeding into a positive-feedback process that ultimately generates exponential or higher power-law growth, at other times by feeding into a form of negative feedback that damps any further acceleration of technological progress. Part IV's illustrative analysis of U.S. patent count data demonstrates how the growth in a given measure of technological progress can pass through distinct phases of behavior.

But this Article does more than provide lessons in contingency and complication. Consistent with expectations under Part II's model, the cumulative U.S. utility-patent count appears to have exhibited substantially constant power-law behavior over decades-long time scales. Part II's model also provides a new framework for policy analysis by giving rise to a multiplicity of dynamic-elasticity or double-ratio tests for evaluating the desirability of policy changes. These tests turn not on a single ratio of raw magnitudes, but instead on a comparison of two ratios, with each ratio representing a percentage change in a push or drag parameter. Because different technology and industry sectors likely have different non-patent-generated pushes and drags, such double-ratio tests amplify reasons to believe that a policy mechanism like patent law will have disparate effects for different technologies and industries. Double-ratio tests thus suggest that observed differences between different technology sectors' policy sensitivities are even more fundamental than previously appreciated.

Further, the conditions for applying double-ratio tests highlight a further implication of Part II's model: a particular policy's success in promoting progress can turn not only on the post-policy and pre-policy values of various push and drag parameters, but also on where the relevant technology sits along a trajectory of technological progress. The current amount of accumulated knowledge or the current speed of progress can significantly affect which results of a policy change have a dominant role in determining the policy's desirability.

Double-ratio tests can also suggest rules of thumb. For example, when technological development is characterized by enormous non-patent-related drag — perhaps due to high regulatory, educational, or disruption costs for introducing new technologies — a policy instrument that contributes incrementally to drag parameters but that also contributes nonnegligibly to push parameters will often accelerate progress. On the other hand, for technological situations in which non-patent-related drag is comparatively small, a policy instrument like patents that tends to add nontrivially to both pushes and drags might be much more likely to act as a net impediment, rather than a net stimulant, to innovation. Such rules of thumb might help explain significant industry-based differences in views about patents' usefulness. Companies experiencing high-drag conditions, such as cash-strapped startups and regulation-saddled pharmaceutical companies, have commonly championed strong patent rights. Heavyweights of otherwise comparatively low-drag information technologies have been more skeptical.

Part II's model thus provides policymakers with useful insights into (1) the potential for different behavioral regimes for innovative progress, (2) the vulnerability of such behavioral regimes to change over time, (3) the ways in which a particular policy can contribute to push and drag, and (4) the existence of dynamic-elasticity or double-ratio tests turning on the comparison of a policy's percentage effects on push and drag parameters. On the other hand, Part II's model presently is far from providing reliable quantitative predictions.[122] Even if Part II's model is the right one to develop such predictions, determination of input values for its push and drag parameters will likely be a nontrivial task. Even for a carefully chosen set of technologies under well-controlled circumstances, derivation of precise relations between particular policies and the values of push and drag parameters will likely be difficult. Systematic characterization of how tuning patent law's various "levers"[123] — for example, gatekeeping requirements such as nonobviousness and enablement, or patent-scope or remedies doctrines such as the doctrine of equivalents — affects push or drag parameters constitutes a project of its own. In the embryonic study of innovation dynamics, there remains much room for advance.

122. *Cf.* SOLOW, *supra* note 40, at 71 ("Any theory that says something about the real world is likely to have implications for policy. But it is only good sense to realize that an abstract theory, like the one I have been developing, can only say abstract things about economic policy.").

123. *See generally* Burk & Lemley, *supra* note 115 (discussing patent "levers").

VI. APPENDIX

A. Behavior at the Top of Two Example S-Shaped Curves

In Part III.A of the main text, *S*-shaped curves are described as resulting from the following differential equation:

$$dv/dt = \alpha + \beta x - \mu - [\gamma(x)]v \qquad \text{(Eq. 36)},$$

where $\gamma(x)$ assumes the functional forms specified by Equations 29 and 30, respectively. In the former case, x is described by a solution that converges to a finite value, x_{max}, with the difference between x_{max} and x decaying exponentially with time in the limit where $(x_{max} - x)/x_{max} << 1$.[124] In the latter case, x continues to grow indefinitely according to a power law, at least to leading order at large times.[125] These leading-order behaviors are derived below.

1. Asymptotic Behavior in the Vicinity of a Technological "Wall"

Equation 29 provides that $\gamma(x) = \gamma_0/[1 - (x/x_{max})^q]$, where $q > 0$. For values of x sufficiently close to x_{max}, one can use the formula $x = x_{max}(1 - f)$, where $f << 1$. Hence, $v = -x_{max}(df/dt)$, and $dv/dt = -x_{max}(d^2f/dt^2)$. Further, $\gamma(x) = \gamma_0/[1 - (1 - f)^q]$. For $qf << 1$, $(1 - f)^q \approx 1 - qf$. Hence, $\gamma(x) \approx \gamma_0/qf$. Substituting into Equation 36, one obtains:

$$-f(d^2f/dt^2) \approx (\alpha/x_{max} + \beta - \mu/x_{max})f - \beta f^2 + (\gamma_0/q)(df/dt) \qquad \text{(Eq. 37)}.$$

Because f, df/dt, and d^2f/dt^2 are all expected to converge to zero as the "wall" at x_{max} is approached, second-order terms in these quantities, $f(d^2f/dt^2)$ and $-\beta x_{max} f^2$, can be neglected in deriving leading-order behavior. The resulting leading-order differential equation is $df/dt \approx -\zeta f$, where $\zeta = q(\alpha + \beta x_{max} - \mu)/\gamma_0 x_{max}$. This differential equation yields the solution $f \approx c_5 e^{-\zeta t}$, where c_5 is a constant.

2. Large-Time Behavior in the Presence of a Technological "Bottleneck"

Equation 30 provides that $\gamma(x) = \gamma_0[1 + (x/x_c)^q]$, where $q > 0$. Suppose that, for large times where $(x/x_c)^q >> 1$ and $\beta x >> (\alpha - \mu)$, x grows exponentially with time: $x \approx ce^{\lambda t}$. Then, to leading order, the $[\gamma(x)]v$-term in Equation 36 grows exponentially in proportion to

124. *See supra* text accompanying note 91.
125. *See supra* text accompanying note 93.

$e^{(q+1)\lambda t}$. No other term in Equation 36 can match this level of exponential growth. Hence, as long as Equation 36 holds, the supposition of exponential growth cannot be correct.

Suppose instead that leading-order growth proceeds according to a power law: $x \approx ct^z$, where z is a positive real number. In accordance with this power-law behavior, $v \approx czt^{z-1}$, and $dv/dt \approx cz(z-1)t^{z-2}$. To leading order in time t, Equation 36 then reduces to $\beta ct^z \approx \gamma_0(c/x_c)^q czt^{(q+1)z-1}$. This relation holds for $z = 1/q$ and $c = x_c(q\beta/\gamma_0)^{1/q}$. Consequently, there is a leading-order solution to Equation 36 in the large-time limit discussed: namely, $x/x_c \approx (\varsigma t)^{1/q}$ with $\varsigma = q\beta/\gamma_0$.

B. Supplemental Dynamic-Elasticity Tests

1. $\Delta\beta/\beta > \Delta\gamma/\gamma$ in a Large-Time Limit for Eq. 6

For large times t where Equation 12 applies, $x \approx c_1 e^{\kappa t}$. Given the exponential dependence of this leading-order behavior on κt and the only linear dependence on c_1, it might be considered prima facie obvious that, at least in a large-time limit, the leading-order test for whether a policy change increases the large-time value of x is whether that policy change increases κ.

More detailed analysis confirms this intuition. Take Δx, Δc_1, and $\Delta\kappa$ to represent the changes in x, c_1, and κ, respectively, due to the policy change. Then, the ratio of (a) the amount of cumulative technological progress under the policy change to (b) the amount of such progress without the policy change is approximately given by $(x + \Delta x)/x = (c_1 + \Delta c_1)e^{(\kappa+\Delta\kappa)t}/c_1 e^{\kappa t} = [(c_1 + \Delta c_1)/c_1]e^{(\Delta\kappa)t}$. The policy change increases the amount of cumulative technological progress if and only if $(x + \Delta x)/x > 1$. Thus, to leading order, the policy change increases the amount of technological progress expected in the large-time limit if and only if $[(c_1 + \Delta c_1)/c_1]e^{(\Delta\kappa)t} > 1$, which is equivalent to the condition $\Delta\kappa > (1/t)\ln[c_1/(c_1 + \Delta c_1)]$. As the time t goes to infinity ($t \to \infty$) — or more generally, for $t >> \ln[c_1/(c_1 + \Delta c_1)]$ — the latter condition can be treated as approximately equivalent to $\Delta\kappa > 0$.

What is $\Delta\kappa$ in terms of parameters such as α, β, and γ? From Equation 9, $\kappa = (1/2)[(\gamma^2 + 4\beta)^{1/2} - \gamma]$. In a large-$\beta$ limit where $2\beta^{1/2} >> \gamma$, it follows that $\kappa \approx \beta^{1/2}$ to leading order. Consequently, in this limit, the leading-order condition for a policy change that increases long-term technological process is $\Delta\beta > 0$. In this large-β limit, positive feedback so dominates long-term behavior that the leading-order question is whether the policy change further increases that feedback, without need for initial reference to the effect of the policy change on γ.

On the other hand, in a small-β limit where $4\beta << \gamma^2$, κ can be approximated by $\kappa \approx \beta/\gamma$. The question of whether a policy change increases κ is then directly analogous to the question of whether a policy change increases the ratio on the right-hand side of Equation 34, except with β taking the place of α' in Equation 34. Substituting accordingly for α' in Equation 35, one obtains the dynamic-elasticity or double-ratio test $\Delta\beta/\beta > \Delta\gamma/\gamma$ indicating when a policy change can be expected to promote progress at large times.

2. $\Delta\beta/\beta > \Delta\gamma/\gamma$ in a Large-Time Limit for Eq. 14

Equation 15 describes the leading-order behavior for cumulative technological progress x in a large-time limit where Equation 14 applies: $x \approx c_3 t^{1/(1-\varepsilon)}$. Assuming that the policy change in question does not alter the exponent ε for the positive-feedback term βx^{ε}, one sees that, to leading order at large times, the change in x due to the policy change is given by the relation $\Delta x \approx (\Delta c_3)t^{1/(1-\varepsilon)}$. Consequently, the change in x is greater than zero if Δc_3, the change in c_3, is greater than zero. The leading-order policy question is whether the policy change increases c_3. Equation 17 indicates that $c_3 = [\beta(1-\varepsilon)/\gamma]^{1/(1-\varepsilon)}$. Hence, where the exponent ε is unaffected by the policy change, the question of whether the policy change increases c_3 is equivalent to the question of whether the policy change increases the ratio β/γ. As seen in the last paragraph of Part VI.B.1, this formulation of the leading-order question yields the ratio test $\Delta\beta/\beta > \Delta\gamma/\gamma$.

3. $\Delta\alpha'/\alpha' > \Delta\xi/\xi$ in a Large-Time Limit for Eq. 18 with $\eta = 1$ and $\gamma/[2(\alpha'\xi)^{1/2}] << 1$

Equation 24 indicates that, in a large-time limit for Equation 18, the speed v of technological progress becomes essentially equal to a terminal velocity v_T, where $v_T = (\varphi - \gamma)/(2\xi)$. Hence, as with the analysis that yielded Equation 35 in Part IV.B, the leading-order question for large times during which Equation 18 is applicable reduces to a question of whether the policy change increases v_T. Recall from Equation 20 that $\varphi = [\gamma^2 + 4\alpha'\xi]^{1/2}$. In a small-$\gamma$ limit where $\gamma/[2(\alpha'\xi)^{1/2}] << 1$, it follows that $\varphi \approx 2(\alpha'\xi)^{1/2}$ and also that $\varphi - \gamma \approx 2(\alpha'\xi)^{1/2}$. Hence, to leading order, $v_T \approx (\alpha'/\xi)^{1/2}$. To a first approximation, the question of whether the policy change increases v_T thus reduces to the question of whether the policy change increases the ratio α'/ξ. This formulation of the leading-order question is directly analogous to the question of whether a policy change increases the ratio on the right-hand side of Equation 34, with ξ taking the place of γ. Substituting accordingly for γ in Equation 35, one obtains the dynamic-elasticity or double-ratio test $\Delta\alpha'/\alpha' > \Delta\xi/\xi$ indicating when a policy change can be expected to promote progress at large times.

4. $\Delta\alpha'/\alpha' > \Delta\gamma/\gamma$ in a Large-Time Limit for Eq. 18 with $\eta = 1$ and $\gamma^2/4\alpha'\xi >> 1$

As in Part VI.B.3 of this Appendix, Equation 24 indicates that, in a large-time limit for Equation 18, the speed v of technological progress becomes essentially equal to a terminal velocity v_T, where $v_T = (\varphi - \gamma)/2\xi$. In a large-$\gamma$ limit where $\gamma^2/4\alpha'\xi >> 1$, it follows that $\varphi \approx \gamma + 2\alpha'\xi/\gamma$ and that, to leading order, $\varphi - \gamma \approx 2\alpha'\xi/\gamma$. Hence, to leading order, $v_T \approx \alpha'/\gamma$. By following the reasoning appearing after Equation 34 in Part IV.C, one arrives at a dynamic-elasticity or double-ratio test identical to that of Equation 35: $\Delta\alpha'/\alpha' > \Delta\gamma/\gamma$.

5. $\Delta\beta/\beta > \Delta\xi/\xi$ in a Large-Time Limit for Eq. 1

In a large-time limit for Equation 1 in which one cannot ignore either the supralinear $\xi v^{1+\eta}$ term or the time dependence of the positive-feedback term βx^{ε}, the cumulative amount of technological progress x has the leading-order behavior indicated by Equation 26: $x \approx c_4 t^{(1+\eta)/(1+\eta-\varepsilon)}$. Under the assumption that the policy change in question does not affect the exponents ε and η, it follows that, to a first approximation, the policy change increases x at large times if the policy change increases c_4. From Equation 28, $c_4 = [(1 + \eta - \varepsilon)/(1 + \eta)]^{(1+\eta)/(1+\eta-\varepsilon)}(\beta/\xi)^{1/(1+\eta-\varepsilon)}$. Recall that, under the general assumptions that $\eta > 0$ and $0 < \varepsilon \leq 1$, $1 + \eta > \varepsilon$ and therefore $1 + \eta - \varepsilon > 0$. Hence, the exponent of the ratio β/ξ in the formula for c_4 is positive. Thus, where the policy change does not affect ε and η, the question of whether the policy change increases c_4 is equivalent to the question of whether the policy change increases the ratio β/ξ. This formulation of the leading-order question is directly analogous to the question of whether a policy change increases the ratio on the right-hand side of Equation 34, with β taking the place of α' and ξ taking the place of γ. Substituting accordingly for α' and γ in Equation 35, one obtains the dynamic-elasticity or double-ratio test $\Delta\beta/\beta > \Delta\xi/\xi$ indicating when a policy change can be expected to promote progress at large times.